

Absence of Replica Symmetry Breaking in a Region of the Phase Diagram of the Ising Spin Glass

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Abstract. We prove that the distribution functions of magnetization and spin glass order parameter coincide on the Nishimori line in the phase diagram of the $\pm J$ Ising model in any dimension. This implies absence of replica symmetry breaking because the distribution function of magnetization consists only of two delta functions, suggesting the same simple structure for the distribution of the spin glass order parameter. It then follows that the mixed (glassy) phase, where ferromagnetic order coexists with a complex phase space structure, should lie below the Nishimori line if it exists at all. We also argue that the AT line that marks the onset of RSB with a continuous distribution of the spin glass order parameter (again, if it exists) would start with an infinite slope from the multicritical point where paramagnetic, ferromagnetic and spin glass phases merge.

INTRODUCTION

The existence and characteristics of the spin glass phase are currently being actively investigated for finite-dimensional random spin systems. Closely related is the problem of the mixed (glassy) ferromagnetic phase. If the mean-field picture applies to finite-dimensional systems, the ferromagnetic phase would split up into two regions, one with a simple structure (the replica-symmetric (RS) phase in the mean-field framework) and the other with a complex phase space (replica-symmetry broken (RSB) state). The boundary between these two phases would be the finite-dimensional counterpart of the AT (de Almeida-Thouless) line found for the infinite-range SK (Sherrington-Kirkpatrick) model.

Investigations of these and related problems are currently being carried out almost exclusively by numerical methods because of a lack of reliable analytical methods. We show in the present contribution that simple symmetry arguments lead to a strong constraint on the possible location and shape of the AT line in the

phase diagram of the Ising spin glass on an arbitrary lattice with arbitrary range of interactions. More precisely, it is possible to prove that there is nothing like RSB on the Nishimori line in the phase diagram and that the AT line, if it ever exists, should start as a vertical line from the multicritical point where paramagnetic, ferromagnetic and spin glass phases merge. Our method is an application of the gauge theory which has been used to derive exact energy and many other exact/rigorous results.

GAUGE THEORY OF THE ISING SPIN GLASS

Let us consider the $\pm J$ Ising model ($S_i = \pm 1$) with the Hamiltonian

$$H = - \sum_{\langle ij \rangle} J_{ij} S_i S_j, \quad (1)$$

where the sum is over pairs of sites on an arbitrary lattice with arbitrary range of interactions. The exchange interactions are quenched random variables with the distribution function

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1 - p)\delta(J_{ij} + J). \quad (2)$$

The Hamiltonian (1) is invariant under gauge transformation

$$J_{ij} \rightarrow J_{ij} \sigma_i \sigma_j, \quad S_i \rightarrow S_i \sigma_i, \quad (3)$$

where σ_i is an arbitrarily fixed Ising spin.

The gauge invariance of the Hamiltonian leads to a number of exact and/or rigorous results [1,2]. For example, consider the internal energy

$$E(K, K_p) = [\langle H \rangle_K]_{K_p}, \quad (4)$$

where the inner triangular brackets denote a thermodynamic average and the outer square brackets represent the configurational average at a fixed value of p . The suffixes denote the temperature and p through $K = \beta J = J/k_B T$ and $K_p = \frac{1}{2} \log p/(1 - p)$. The exact value of the internal energy can be calculated explicitly under the constraint $K_p = K$ to give

$$E(K, K) = -N_B J \tanh K, \quad (5)$$

where N_B is the number of bonds on the lattice under consideration. The constraint $K = K_p$ defines a curve in the phase diagram shown dashed in Figure 1, which is often called the Nishimori line.

There are many other results including an upper bound on the specific heat

$$k_B T^2 C(K, K) \leq \frac{J^2 N_B}{\cosh^2 K}, \quad (6)$$

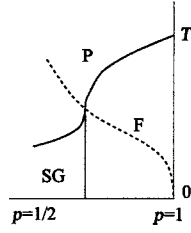


FIGURE 1. Phase diagram of the $\pm J$ model and the Nishimori line (dashed).

a correlation identity

$$[\langle S_i S_j \rangle_K]_K = [\langle S_i S_j \rangle_K^2]_K, \quad (7)$$

and a correlation inequality

$$[\langle S_i S_j \rangle_K]_{K_p} \leq [|\langle S_i S_j \rangle_{K_p}|]_{K_p}. \quad (8)$$

The correlation identity (7) means that the Nishimori line ($K_p = K$) avoids the spin glass phase because the left-hand side reduces to the square of the ferromagnetic long range order m^2 as the distance between i and j tends to infinity whereas the right-hand side approaches the square of the spin glass order parameter q^2 in the same limit. The spin glass phase is characterized by $q > 0, m = 0$ which is inconsistent with the relation $m = \pm q$. The correlation inequality (8) places a constraint on the possible shape of the boundary between the ferromagnetic and spin glass phases: This boundary can either be vertical or re-entrant in the phase diagram. The ferromagnetic phase is not allowed to lie below the spin glass phase [1,2].

DISTRIBUTION OF ORDER PARAMETERS

The gauge theory summarized above can be applied to derive an identity between the distribution functions of the ferromagnetic and spin glass order parameters on the Nishimori line [2,3].

Definitions and the main result

The distribution function of magnetization is defined as

$$P_m(x; K) = \left[\frac{\sum_S \delta(x - N^{-1} \sum_i S_i) e^{-\beta H(S)}}{\sum_S e^{-\beta H(S)}} \right]_{K_p}. \quad (9)$$

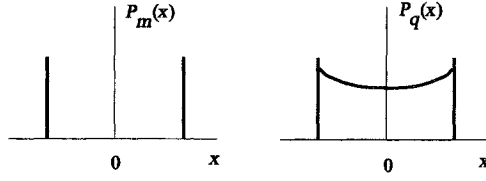


FIGURE 2. Distribution of magnetization (left) and the SG order parameter (right).

To define the distribution function of the spin glass order parameter, it is convenient to introduce two replicas of the same system with spins $\{S_i\}$ and $\{\sigma_i\}$ and the couplings K_1 and K_2 , respectively:

$$P_q(x; K_1, K_2) = \left[\frac{\sum_S \sum_\sigma \delta(x - N^{-1} \sum_i S_i \sigma_i) e^{-\beta_1 H(S)} e^{-\beta_2 H(\sigma)}}{\sum_S \sum_\sigma e^{-\beta_1 H(S)} e^{-\beta_2 H(\sigma)}} \right]_{K_p}, \quad (10)$$

where $K_1 = \beta_1 J$ and $K_2 = \beta_2 J$. The distribution of the spin glass order parameter is defined in terms of two replicas with the same temperature, $P_q(x; K) \equiv P_q(x; K, K)$. Note that the distributions $P_m(x; K)$, $P_q(x; K, K_p)$ and $P_q(x; K)$ are all functions of p as well.

It is well established that the distribution of magnetization has a simple structure with two delta functions as depicted in the left part of Figure 2. The spin glass order parameter, on the other hand, has a more complex structure. For example, if there exists full RSB, the distribution has a continuous part as depicted schematically in the right part of Figure 2. Or, it may have delta peaks at a few additional locations in the case of finite-step RSB. In any case, the complexity of the system is closely related to the structure of the distribution function $P_q(x)$ being different from that of $P_m(x)$.

Our main result is the following relation:

$$P_m(x; K) = P_q(x; K, K_p) \quad (11)$$

which holds for any combination of x , K and K_p . In particular, if we set $K_p = K$, Equation (11) reduces to the relation between the usual distribution functions of magnetization and spin glass order parameter:

$$P_m(x; K_p) = P_q(x; K_p). \quad (12)$$

This immediately proves that the distribution function of the spin glass order parameter on the right-hand side has a simple structure represented by the magnetization distribution on the left-hand side if $K = K_p$.

The simplicity of the exact energy (5) and a few other results [2,1] suggest that the phase space of the system would not be complicated. The above result is regarded as definite evidence to support this anticipation.

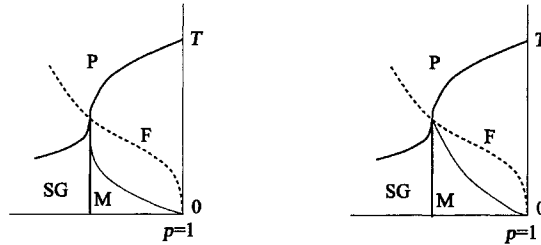


FIGURE 3. Phase diagrams compatible (left) and incompatible (right) with the relations proved in the text when the mixed phase is characterized by a continuous part in $P_q(x)$.

It also follows from differentiation of Equation (11) that the derivatives of the distribution functions $P_m(x; K)$ and $P_q(x; K)$ with respect to K and p vanish when $K = K_p$:

$$\frac{\partial}{\partial K} P_q(x; K) = 0, \quad \frac{\partial}{\partial p} P_q(x; K) = 0 \quad (13)$$

for almost all x when $K = K_p$. These relations have been derived from the fact that the distribution of magnetization $P_m(x; K)$ consists of just two delta functions and therefore its derivatives with respect to the parameters K and p vanish for almost all x . Equation (13) then follows from (11).

The relation (13) shows that the distribution function of the spin glass order parameter remains vanishing at almost all x if the parameters K and p deviate infinitesimally from their values satisfying $K = K_p$. Thus the structure of $P_q(x; K)$ does not develop a continuous part under an infinitesimal deviation from the Nishimori line. An interesting consequence is that the mixed phase with a continuous part in $P_q(x)$, if it starts to exist at the multicritical point, does not have a finite extension under a very small deviation from the multicritical point. Stated otherwise, the phase boundaries (the AT line and the boundary between the mixed and spin glass phases) should merge smoothly as they approach the multicritical point as depicted on the left part of Fig. 3. A natural generic situation would be that both of these boundaries are vertical around the multicritical point as in the case of the SK model. It should be noted that this argument does not apply when the RSB in the mixed phase is characterized by several delta functions, in which case a phase boundary with a structure as sketched on the right-hand side of Fig. 3 does not violate (13).

Outline of the proof

The proof of the relation (11) is not very difficult [3]. We start from writing out the configurational average of the distribution function of magnetization

$$P_m(x; K) = \frac{1}{(2 \cosh K_p)^{N_B}} \sum_{\{\tau_{ij}=\pm 1\}} e^{K_p \sum \tau_{ij}} \frac{\sum_S \delta(x - N^{-1} \sum_i S_i) e^{K \sum \tau_{ij} S_i S_j}}{\sum_S e^{K \sum \tau_{ij} S_i S_j}}. \quad (14)$$

The weight of the configurational average (p or $1 - p$ for each bond) is taken care of by the factor $e^{K_p \sum \tau_{ij}}$ [1,2]. The gauge transformation (3) changes the above equation into the form

$$P_m(x; K) = \frac{1}{(2 \cosh K_p)^{N_B}} \sum_{\{\tau_{ij}=\pm 1\}} e^{K_p \sum \tau_{ij} \sigma_i \sigma_j} \frac{\sum_S \delta(x - N^{-1} \sum_i S_i \sigma_i) e^{K \sum \tau_{ij} S_i S_j}}{\sum_S e^{K \sum \tau_{ij} S_i S_j}}. \quad (15)$$

It is relatively straightforward to see that the right-hand side is equal to $P_q(x; K_p, K)$ by summing it up over all possible $\{\sigma_i\}$, dividing the result by 2^N , and finally inserting the identity $1 = \sum_\sigma e^{K_p \sum \tau_{ij} \sigma_i \sigma_j} / \sum_\sigma e^{K_p \sum \tau_{ij} \sigma_i \sigma_j}$ after the summation symbol over $\{\tau_{ij}\}$.

CONCLUSION

We have proved that the AT line, if it exists, should lie below the Nishimori line in the phase diagram of the $\pm J$ Ising model on an arbitrary lattice. It has also been argued that the mixed phase with a continuous part in the distribution function of the spin glass order parameter, again if it exists, does not have a finite extension around the multicritical point. These results have been derived by a simple application of gauge transformations.

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