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Duality and Multicritical Point of Two-Dimensional Spin Glasses

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Determination of the precise location of the multicritical point and phase boundary is a target of active current research in the theory of spin glasses.[1–11] In this short note we develop a duality argument to predict the location of the multicritical point and the shape of the phase boundary in models of spin glasses on the square lattice.

The first system we treat is a random Z_q model with gauge symmetry which includes the $\pm J$ Ising model and the Potts gauge glass. Following the notation of Wu and Wang,[12] the partition function is

$$Z = \sum_{\{\xi_i\}} \exp\left\{ \sum_{\langle ij \rangle} V(\xi_i - \xi_j + J_{ij}) \right\}. \tag{1}$$

Here, ξ_i $(= 0, 1, \cdots, q-1)$ is the q-state spin variable, J_{ij} $(= 0, 1, \cdots, q-1)$ is the quenched randomness, and $V(\cdot)$ is a periodic function with period q. The sum in the exponent runs over neighbouring sites on the square lattice. It is straightforward to generalize the duality transformation in the Wu–Wang formalism[12] to the random model, and the result is

$$Z = \sum_{\{\eta\}}{}' \exp\left\{ \sum_{\langle ij \rangle} V(\eta_{ij} + J_{ij}) \right\} \tag{2}$$

$$= q^N \sum_{\{\lambda\}}{}' \exp\left\{ \sum_{\langle ij \rangle} \tilde{V}(\lambda_{ij}) + \frac{2\pi \mathrm{i}}{q} \lambda_{ij} J_{ij} \right\}, \tag{3}$$

where η_{ij} denotes $\xi_i - \xi_j$ and λ_{ij} is the bond variable on the dual lattice corresponding to η_{ij} and $\mathrm{e}^{\tilde{V}}$ is the Fourier transform of e^{V}. The prime in the sum indicates that the bond variables η_{ij} and λ_{ij} are constrained such that their sums over a closed loop vanish.[12] Thus the duality transformation consists in replacing the exponential in (2) with that in (3):

$$\mathrm{e}^{V(\eta+J)} \leftrightarrow \mathrm{e}^{\tilde{V}(\lambda) + 2\pi \mathrm{i} \lambda J/q}. \tag{4}$$

Average over quenched randomness is dealt with by the replica method. A generalization of the existing argument[13] suggests that it is convenient to consider the following quantity

$$\alpha = \frac{\displaystyle\sum_{l=0}^{q-1} p_l \left(\sum_{\eta=0}^{q-1} \mathrm{e}^{V(\eta+l)} \right)^n}{\displaystyle\sum_{l=0}^{q-1} p_l \mathrm{e}^{nV(l)}}, \tag{5}$$

where p_l is the probability that J_{ij} takes the value l. It is not difficult to check that the duality transformation (4) changes α into $\tilde{\alpha} = q^n/\alpha$. The subspace defined by $\alpha = q^{n/2}$ in the parameter space $(p_1, p_2, \cdots, T, \cdots)$ is therefore left invariant by the duality transformation: A point in this subspace is transformed into another point in the same subspace although the latter point does not necessarily have real values of Boltzmann weights. In general it is possible to consider complex values of Boltzmann weights (or parameters such as the temperature); the subspace $\alpha = q^{n/2}$ embedded in the complex-valued parameter space maps onto itself by the duality transformation (4). We shall consider the cross section of this subspace with the phase diagram with real parameters, which we shall call the invariant subspace in the following.

One should be careful in identifying the invariant subspace with a critical surface.[14] We discuss this problem later; let us see here what the invariant subspace looks like in the quenched system, which is obtained by taking the limit $n \to 0$ of the relation $\alpha = q^{n/2}$:

$$\sum_{l=0}^{q-1} p_l \log\left(\sum_{\eta=0}^{q-1} \mathrm{e}^{V(\eta+l) - V(l)} \right) = \frac{1}{2} \log q. \tag{6}$$

We now apply (6) to a few examples and discuss the consequences. The first case is the $\pm J$ Ising model $(q = 2)$ with $V(0) = \beta J, V(1) = -\beta J$ and $p_0 = p, p_1 = 1 - p$. The invariant subspace (6) then reads

$$p \log(1 + \mathrm{e}^{-2\beta J}) + (1 - p) \log(1 + \mathrm{e}^{2\beta J}) = \frac{1}{2} \log 2. \tag{7}$$

This curve is plotted in Fig. 1 in the p–T phase diagram. It is observed (and can be confirmed analytically) that the curve reaches the point with minimum p on the Nishimori line[15] $\mathrm{e}^{-2\beta J} = (1 - p)/p$, from which and (7), the minimum point p_c is found to satisfy

$$-p_\mathrm{c} \log p_\mathrm{c} - (1 - p_\mathrm{c}) \log(1 - p_\mathrm{c}) = \frac{\log 2}{2}, \tag{8}$$

or $p_\mathrm{c} = 0.889972$. This coincides with numerical estimates of p at the multicritical point with high precision: 0.8905(5),[1] 0.886(3),[2] 0.8872(8),[3] 0.8906(2)[4] and 0.8907(2).[5]

The next example is the q-state Potts gauge glass with $V(0) = \beta J, V(1) = V(2) = \cdots = V(q-1) = 0$ and $p_0 = 1 - (q-1)p, p_1 = \cdots = p_{q-1} = p$, where we have followed the notation of Jacobsen and Picco.[6] The invariant subspace (6) is, in this case,

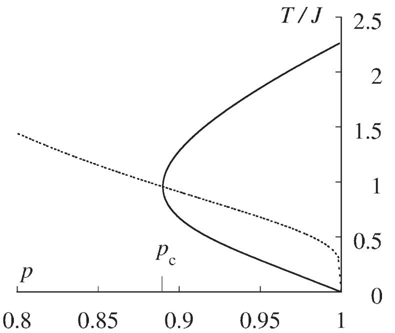

Fig. 1. Invariant subspace of the $\pm J$ Ising model is the curve drawn by the solid line. Shown dashed is the Nishimori line.

$$\{1-(q-1)p\}\log\{1+(q-1)\mathrm{e}^{-\beta J}\} + (q-1)p\log(q-1+\mathrm{e}^{\beta J}) = \frac{\log q}{2}. \tag{9}$$

This formula reduces to the Ising counterpart (7) when $q = 2$ under appropriate changes of energy scale and notation (exchange of p and $1-p$). The curve specified by (9) reaches the extremum point in the p–T phase diagram again on the Nishimori line $p = 1/(\mathrm{e}^{\beta J} + q - 1)$.[16] In particular, for $q = 3$, this extremum point p_c satisfies

$$-(1-2p_c)\log(1-2p_c) - 2p_c\log p_c = \frac{\log 3}{2}, \tag{10}$$

whose numerical value $p_c = 0.0797308$ coincides very well with the location of the multicritical point, 0.079–0.080.[6]

It is possible to apply the same argument to the random Ising model with a general distribution function $P(J_{ij})$ of exchange interactions.[13] The result for the invariant subspace is

$$\int_{-\infty}^{\infty} \mathrm{d}J_{ij}\, P(J_{ij})\, \log(1+\mathrm{e}^{-2\beta J_{ij}}) = \frac{\log 2}{2}. \tag{11}$$

This formula applied to the Gaussian model with $P(J_{ij}) \propto \mathrm{e}^{-(J_{ij}-J_0)^2/2J^2}$ represents a curve in the phase diagram similar to that of the $\pm J$ model in Fig. 1 and again has the smallest J_0/J on the Nishimori line $\beta J^2 = J_0$.[15] This point is at $J_0/J = 1.02177$ which is very close to the multicritical point evaluated numerically (Horie, Nemoto, Hukushima and Ozeki, private communication).

What do all these results mean? One possibility is that the exact locations of multicritical points have been derived for the above models. The present argument using the subspace invariant under duality is, in general, not guaranteed to give the exact phase boundary or multicritical point. Nevertheless we have several reasons to conjecture that our result may be exact, in particular for the multicritical point.

The first reason is that the invariant subspace $\alpha = q^{n/2}$ coincides with the exact phase boundary for the $\pm J$ Ising model in the case of $n = 1$ and $n = 2$.[17] It should be noticed that this coincidence exists only above the multicritical point in the phase diagram for the $n = 2$ case, which may be related to the strange shape of the curve below the multicritical point in Fig. 1. The second reason is the impressive agreement with numerical estimates explained above, which seems to us beyond a simple coincidence. The third evidence is the fact that the Nishimori line appears naturally from the present duality formalism. It is well established that the multicritical point is located on the Nishimori line for models with gauge symmetry.[18] The natural emergence of the Nishimori line from arguments without explicit use of gauge transformation strongly suggests that something deep may be hidden behind the scene.

There are of course several problems that deserve special caution. As pointed out by Aharony and Stephen,[14] duality does not yield fixed points of the transformation for random models whereas fixed points of duality are often identical to critical points in non-random systems. Aharony and Stephen thus argued that duality in random systems, unable to identify fixed points, is not a useful tool of analysis. We are suggesting here that an invariant subspace, if not fixed points, may be of some use in random systems.

Another problem is whether or not the whole invariant subspace (6) or (11) coincides with the phase boundary. It is dangerous to accept this identification at least below the multicritical point in the phase diagram because of the $\pm J$ Ising model with $n = 2$ as mentioned above. The origin of disagreement of our result (7) near $p = 1$ with the perturbative calculation of Domany[19] should also be clarified in future investigations.

1) F. D. A. Aarao Reis, S. L. A. de Queiroz and R. R. dos Santos: Phys. Rev. B **60** (1999) 6740.
2) R. R. P. Singh and J. Adler: Phys. Rev. B **54** (1996) 364.
3) Y. Ozeki and N. Ito: J. Phys. A **31** (1998) 5451.
4) A. Honecker, M. Picco and P. Pujol: Phys. Rev. Lett. **87** (2001) 047201.
5) F. Merz and J. T. Chalker: Phys. Rev. B **65** (2002) 054425.
6) J. L. Jacobsen and M. Picco: Phys. Rev. E **65** (2002) 026113.
7) S. Cho and M. P. A. Fisher: Phys. Rev. B **55** (1997) 1025.
8) I. A. Gruzberg, N. Read and A. W. W. Ludwig: Phys. Rev. B **63** (2001) 104422.
9) N. Read and A. W. W. Ludwig: Phys. Rev. B **63** (2000) 024404.
10) T. Senthil and M. P. A. Fisher: Phys. Rev. B **61** (2000) 9690.
11) E. S. Sørensen, M. J. P. Gingras and D. A. Huse: Europhys. Lett. **44** (1998) 504.
12) F. Y. Wu and Y. K. Wang: J. Math. Phys. **17** (1976) 439.
13) H. Nishimori: J. Phys. C **12** (1979) L905.
14) A. Aharony and M. J. Stephen: J. Phys. C **13** (1980) L407.
15) H. Nishimori: Prog. Theor. Phys. **66** (1981) 1169.
16) H. Nishimori and M. J. Stephen: Phys. Rev. B **27** (1983) 5644.
17) A. George, D. Hansel, P. Le Doussal and J. M. Maillard: J. Phys. (Paris) **48** (1987) 1.
18) H. Nishimori: *Statistical Physics of Spin Glasses and Information Processing: An Introduction* (Oxford Univ. Press, Oxford, 2001).
19) E. Domany: J. Phys. C **12** (1979) L119.