

Statistical Mechanics of Phase Unwrapping Problem by the Q -Ising Model

Y. Saika* and H. Nishimori†

*Department of Electrical Engineering, Wakayama National College of Technology, 77 Noshima, Nada, Gobo, Wakayama 644-0023, Japan

†Department of Physics, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro, Tokyo 152-8551, Japan

Abstract. We construct the statistical mechanical formulation for the problem of phase unwrapping, appearing in adaptive optics. We estimate the performance of our method using the replica theory and the time-dependent Ginzburg Landau theory for the infinite-range model. The replica theory clarifies that our method works well if we select appropriate model of the prior. Then, the time-dependent Ginzburg Landau theory estimates the dynamical property of the simulated annealing. These results are qualitatively confirmed by the Monte Carlo simulations for the realistic model.

INTRODUCTION

In recent years, a lot of researchers in statistical physics have been working on problem related to information processing [1-5], such as image analysis, statistics for spatial data, and Markov random fields. Various methods of statistical mechanics, such as the mean-field theory [6], have been applied to problems of image restoration and error-correcting codes.

In adaptive optics [7-9], phases often carry information through noisy transmission. Retrieving phases is therefore necessary using the measured phase differences in the principle interval $[-\pi, \pi)$. Especially this problem becomes difficult due to the under-sampling, even if no noise is introduced into measured variables.

In the present research, we formulate the problem of phase reconstruction on the basis of the statistical mechanics of the Q -Ising model. Further, we estimate the performance of our method using the replica theory for the infinite-range model and the Monte Carlo simulation for the realistic model. The replica theory derives the result that our method due to the MPM estimate works well irrespective of the initial condition when the Nyquist condition holds. This result is confirmed by Monte Carlo simulation for a typical phase pattern in adaptive optics. If the Nyquist condition does not completely hold at every

sampling point, the Monte Carlo simulation derives the result that our method using the simulated annealing works well when we start from appropriate initial phase pattern.

GENERAL FORMULATION

We first show the statistical mechanical formulation for phase reconstruction using the Q -Ising model. Here we use the language of adaptive optics.

First, we consider a set of original phase pattern $\{\xi_{(x,y)}\}$, which is generated with the probability $P(\{\xi\})$. Here $\xi_{(x,y)} = -R/2 + kR/Q$, $k = 1, \dots, Q$, $x, y = 1, 2, \dots$. Then, the complex phase difference expressed as $\hat{\xi}_{(x,y)} \hat{\xi}_{(x',y')}^* = \exp(-j(\xi_{(x,y)} - \xi_{(x',y')}))$ is transmitted through the noisy channel and is then rewritten into $\hat{J}_{(x,y|x',y')} = |J_{(x,y|x',y')}| \exp(-jA_{(x,y|x',y')})$ with the conditional probability:

$$P(\{J\} | \{\xi\}) = \frac{1}{(2\pi J^2)^{N_B}} \exp\left(-\frac{1}{2J^2} \sum_{n,n'} \hat{J}_{(x,y|x',y')} - J_0 \hat{\xi}_{(x,y)} \hat{\xi}_{(x',y')}^* \right)^2 \quad (1)$$

where $|A_{(x,y|x',y')}| < \pi$. Then J_0 and J^2 are the mean and variance of the Gaussian noise with each other.

Next, we retrieve phases by the MPM estimate by using the system $\{z_{(x,y)}\}$ where $z_{(x,y)} = -R/2 + k/RQ$, $k = 1, \dots, Q$, if observed data are considered to be sufficiently sampled. Then, the retrieved information is given by

$$\hat{z}_{(x,y)} = \arg \max_{z_{(x,y)}} \text{Tr} P(\{z\} | \{J\}). \quad (2)$$

Here this posterior probability is given by the Bayes formula:

$$P(\{z\} | \{J\}) \propto P(\{z\})P(\{J\} | \{z\}) \quad (3)$$

where we assume the models of the true prior and the noise probability as

$$P(\{z\}) \propto \exp\left(-\frac{1}{T_m} \sum \left((z_{(x+1,y)} - z_{(x,y)})^2 + (z_{(x-1,y)} - z_{(x,y)})^2 \right)\right), \quad (4)$$

$$P(\{J\} | \{z\}) \propto \exp\left(-\frac{\beta J T_s}{T_m} \sum_{n,n'} \cos(A_{(x,y|x',y')} - z_{(x,y)} + z_{(x',y')})\right). \quad (5)$$

On the other hand, when measured data are considered not to be sufficiently sampled, we use the initial phase pattern constructed by the following procedure:

$$z'_{(x,y)} = \frac{1}{2} (u_{(x,y)} + v_{(x,y)}), \quad (6)$$

$$u(x, y) = A_{(x-1,y|x,y)} + 2\pi m_{(x,y)} - z'_{(x-1,y)} + z'_{(x-2,y)}, \quad (7)$$

$$v(x, y) = A_{(x,y-1|x,y)} + 2\pi n_{(x,y)} - z'_{(x,y-1)} + z'_{(x,y-2)}, \quad (8)$$

where $m_{(x,y)}$ and $n_{(x,y)}$ are determined so as to minimize $(\partial_x z_{(x,y)} - \partial_x z_{(x-1,y)})^2$ and $(\partial_y z_{(x,y)} - \partial_y z_{(x,y-1)})^2$.

Here $\partial_x z_{(x,y)} = z_{(x,y)} - z_{(x-1,y)}$ and $\partial_y z_{(x,y)} = z_{(x,y)} - z_{(x,y-1)}$.

Using this initial pattern, we retrieve phases by the conventional simulated annealing. Here the retrieved phase is given by

$$\hat{z}_{(x,y)} = \arg \max_{\{z\}} P(\{z\} | \{J\}) \quad (9)$$

When we estimate the performance of our method, we use the sample average of the mean square error between original and the retrieved phase patterns as

$$\hat{\sigma} = \left[\sum_{(x,y)} \frac{1}{(2\pi L)^2} \left(\xi_{(x,y)} - \hat{z}_{(x,y)} \right)^2 \right]_{\xi,J} \quad (10)$$

INFINITE-RANGE MODEL

Replica Symmetric Theory

The replica theory for the infinite-range model is useful to give a guide to the qualitative understanding of system properties. Here we evaluate how the mean square error depends on the parameter T_m . We first note three assumptions should be introduced when we use the replica theory: (1) the large number limit of lattice/sampling points; (2) the infinite-range versions of the true prior and the noise probability as

$$P(\{\xi\}) \propto \exp\left(-\frac{1}{NT_s} \sum_{i \neq j} (\xi_i - \xi_j)^2 - h \sum_i \xi_i\right), \quad (11)$$

$$P(\{J\} | \{\xi\}) = \frac{N}{2\pi J^2} \exp\left(-\frac{1}{2J^2} \left| J_y - \frac{J_0}{N} \sum_i \xi_i \right|^2\right) \quad (12)$$

where i and j are the sites on the grid and N is the number of sites; (3) the replica theory within the replica symmetric assumption. Next, under these assumptions, we derive the self-consistent equations for m_0, m_x, m_y, q and m due to the saddle-point conditions of the free energy. As is shown in Fig. 1, with the use of the solutions of these equations, we obtain the result that the mean square error $\hat{\sigma}$ takes

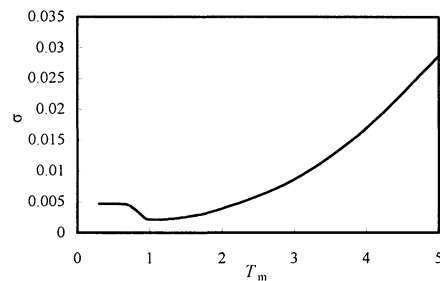


FIGURE 1. The mean square error σ as a function of T_m using the replica theory. Other parameters are set as $T_s = 1.0$, $J_0 = 1.0$, $J = 0.3$, $\beta_J = 1.0$, $h = 0.06$.

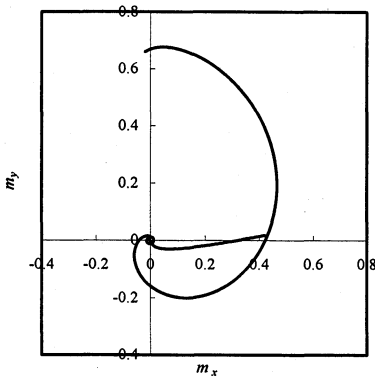


FIGURE 2. The time evolution of the macroscopic state on the $m_x - m_y$ plane due to the TDGL theory. Here $T_s = 1.0$, $J_0 = 1.0$, $J = 0.3$, $\beta_J = 1.0$, $h = 0.06$.

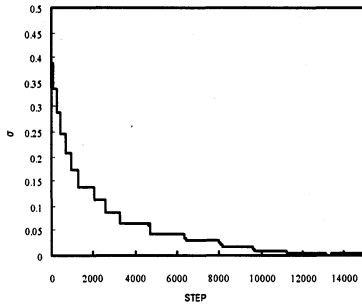


FIGURE 3. The time evolution of the mean square error σ obtained by the TDGL theory where $T_s = 1$, $h_0 = 0.05$, $J_0 = 1$, $J = 0.3$, $\beta_J = 0.5$, $h = 1$

its minimum when we appropriately tune parameter T_m , where other parameters are set as $J_0 = 1$, $J = 0.3$, $\beta_J = 0.5$, $T_s = 1$. In this case, the observed data is considered not to be sampled sufficiently. This result also means that it is not necessary to lower temperature T_m than $T_m < 1.0$ when we retrieve phases using the simulated annealing.

Time-Dependent Ginzburg Landau Theory for Infinite-Range Model

Next, we qualitatively estimate the dynamical property of our method due to the simulated annealing using the time-dependent Ginzburg Landau theory for the infinite-range model. Here, we evaluate the time evolution of macroscopic states m_0 , m_x , m_y , q , m and the mean square error $\hat{\sigma}$.

As is shown in Fig. 2 and Fig. 3, the TDGL theory clarifies the result that the macroscopic state closes to the optimal state by going around the origin on the $m_x - m_y$ plane through the annealing procedure.

MONTE CARLO SIMULATION

MPM Estimate

In order to confirm the above results of the infinite-range model, we carry out here Monte Carlo simulations for the two-dimensional realistic model.

Here, we estimate the performance by our method for the typical phase pattern:

$$\theta^{\xi}(x,y) = A_s \exp\left\{-\frac{1}{2J^2}\left\{\left(x-\frac{L}{2}\right)^2 + \left(y-\frac{L}{2}\right)^2\right\}\right\} \quad (13)$$

in adaptive optics. This pattern is corrupted by the Gaussian noise with a mean $\hat{\xi}(x,y)\hat{\xi}^*(x',y')$ and a variance $J = 0.3$. To estimate the performance of our method, we use the mean square error $\hat{\sigma}$ which is averaged over 10 patterns corrupted from the original pattern.

We evaluate how the mean square error $\hat{\sigma}$ depends on T_m when the Nyquist condition holds. Shown in Fig. 4, we clarify that the mean square error σ gradually closes to its minimum as T_m decreases and that it takes its minimum value $T_m < 2.0$ where $L = 64$, $A_s = 10.0$, $J_s = 6.0$.

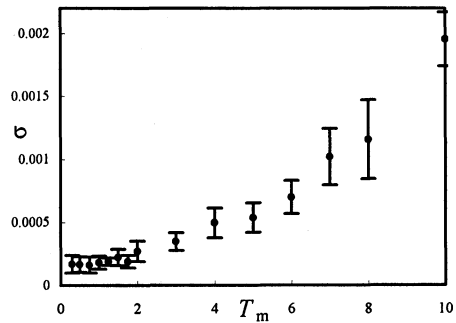


FIGURE 4. The mean square error σ as a function of T_m using the Monte Carlo simulation for typical pattern when $T_s = 1.0$, $J_0 = 1.0$, $J = 0.3$, $\beta_J = 1.0$, $h = 0.06$.

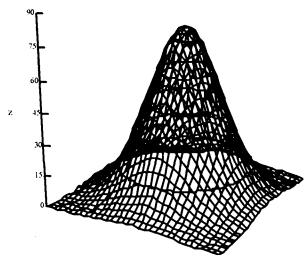


FIGURE 5. The retrieved pattern by the Monte Carlo simulation for a typical one in adaptive optics where $T_s = 1.0$, $J_0 = 1.0$, $J = 0.3$, $\beta_j = 1.0$, $h = 0.06$.

Simulated Annealing

Next, when the Nyquist condition does not hold at every sampling point, that is, the sample pattern we use is given as $L = 32$, $A_s = 90$, $J_s = 6.0$ in (13), we estimate the performance of our method by the Monte Carlo simulation. Figure 5 shows the retrieved phases by our method.

SUMMARY AND DISCUSSION

In the previous chapters, we construct the statistical mechanical formulation for the problem of phase reconstruction by the Q -Ising model. We then estimate the performance of our method using the replica theory and the time-dependent Ginzburg Landau theory for the infinite-range model. We show that our method works well when we tune appropriate parameters and that the phases are gradually retrieved through the annealing procedure irrespective of the initial condition. We estimate the performance of our method for the two-dimensional realistic model using the Monte Carlo simulations. When the Nyquist condition holds, phases are retrieved irrespective of the initial condition. On the other hand, when the Nyquist condition does not hold completely at every sampling point, our method works well when we start from the appropriate initial condition.

ACKNOWLEDGMENTS

We would like to thank Prof. Y. Iba for his fruitful discussions and comments. One of the authors (YS) would like to thank Dr. M. Okada, Dr. T. Aonishi and Dr. M. Inoue for fruitful discussions.

1. Nishimori, H., *Statistical Physics of Spin Glasses and Information Processing: An Introduction*, Oxford University Press, Oxford, 2001, pp74-213.
2. Geman, S., and Geman, D., *IEEE Trans. PAMI* **6**, 721-741 (1984).
3. Surlas, N., *Nature* **339**, 693-695 (1989).
4. Rujan, P., *Phys. Rev. Lett.* **70**, 2968-2971 (1993).
5. Pryce, J. M., and Bruce, D. B., *J. Phys. A* **28**, 511-532 (1995).
6. Tanaka, K., and Morita, T., *Phys. Lett.* **203A**, 122-128 (1995).
7. Fried, D. L., *J. Opt. Soc. Am.* **67**, 370-375 (1976).
8. Takajo, H., and Takahashi, T., *J. Opt. Soc. Am. A* **5**, 416-425 (1986).
9. Nico, G., Palubinskas, G., and Datcu, M., *IEEE Trans. Signal Processing* **48**, 2545-2555 (2000).