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Duality of the Random Model and the Quantum Toric Code

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We study the phase diagrams of random models with bimodal randomness, especially the random bond Ising model and the random plaquette gauge model from the viewpoint of quantum information theory. Using Fourier transformation in conjunction with the replica trick, we show that these models have a common structure under duality transformation. This observation enables us to derive a conjecture on the exact location of the multicritical point and lead the accuracy threshold of the toric code, which is one of the topological quantum memories.

KEYWORDS: quantum information, error correcting code, spin glass, lattice gauge theory, phase transition, duality

1. Introduction

Since the invention of the computational method using quantum states, it has been a challenge to establish a storage method of quantum information because of the instability of quantum states.

Quantum error correcting code is one of the answers to this question. Using this code, errors in qubit systems can be removed with the aid of redundancy. However most of the quantum error correcting codes have a problem in a practical sense; they do not work properly even if error rate is not so large.

Kitaev[1] proposed a new error correcting code called *toric code*. This code is defined on a torus and qubits reside on the bonds of the lattice. The advantage of this code is its robustness against noise and this feature is due to the homological nontriviality of the torus. Thus it is important to determine the accuracy threshold of this code, or the noise rate threshold for a proper error correction from the viewpoint of device design.

Recently it was pointed out that the phase transition in spin glass systems has a relation with the property of the toric code. More precisely, the accuracy threshold of the toric code can be estimated from the location of the multicritical point of the $3D$ random plaquette gauge model (RPGM), which is one of the discrete gauge glass models.[2] Using this correspondence, the accuracy threshold can be estimated numerically. However it is not easy to determine the location of the multicritical point for the RPGM with high precision because of the difficulty of numerical analysis.

In this article we propose an alternative way to determine the multicritical point, which makes use of the dual structure between random models. It was shown that the two-dimensional random bond Ising model (RBIM) has a dual structure from the analogy with the famous Kramers-Wannier's duality for the $2D$ square lattice Ising model.[3] We generalize this idea of the duality to general random Z_2 spin models with bimodal randomness and show that the $3D$ RPGM and the $3D$ RBIM on the cubic lattice are mutually dual. We will check the validity of the generalized duality in some cases and estimate the multicritical point of the $3D$ RPGM from the one of the $3D$ RBIM, the latter of which has been investigated numerically with high precision.

In addition to the accuracy threshold of the toric code, another aspect should be noticed as well; the duality may be useful for investigating phase diagram of random spin models analytically, which is quite difficult in general.

2. Toric code

Let us review the toric code and its relation with the RPGM. We restrict our attention to the code on the $2D$ torus though this code can be generalized to arbitrary manifolds with periodic boundary conditions (e.g. the $4D$ torus[2,4]).

The $2D$ toric code is defined on the square lattice on the $2D$ torus and qubits reside on bonds (Fig.1). States of a single qubit on a bond l (denoted by $|\Psi_l\rangle$) are represented by the expansion over two orthogonal bases as

$$|\Psi_l\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}, \tag{1}$$

where a, b are appropriate coefficients. Here we represent the two orthogonal bases by 2-vector. A quantum state of the whole system $|\Psi\rangle$ is given by the direct product of these states, $|\Psi\rangle = \bigotimes_l |\Psi_l\rangle$.

This system suffers from errors caused by decoherence or interaction with environment, and such errors should be removed for storage of quantum information. Stable errors to be removed are expressed in terms of two errors; *bit-flip error* which flips the qubit and *phase error* to change the relative phase between two bases. They are represented by the operations of the Pauli matrices, $\sigma_l^x|\Psi\rangle$ (bit-flip error) and $\sigma_l^z|\Psi\rangle$ (phase error), where $\sigma_l^{x,z}$ are the Pauli matrices operating on a state of bond l.

For removal of these errors, let us define two check operators,

$$X_i = \bigotimes_{i\in\partial l} \sigma_l^x, \quad Z_P = \bigotimes_{l\in\partial P} \sigma_l^z, \tag{2}$$

where X_i, Z_P are defined on each site and plaquette, respectively (Fig.1). As can be confirmed easily, all check operators at any sites or plaquettes commute with each other, $[X_i, Z_P] = 0$. Hence we can define simultaneous

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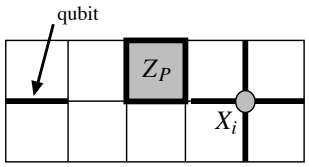

Fig. 1. Qubit and check operators.

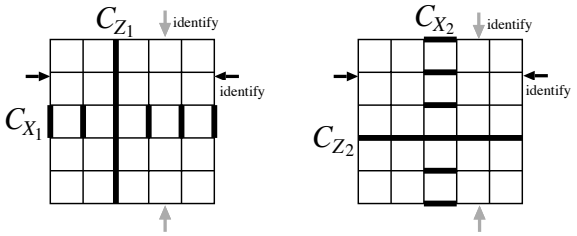

Fig. 2. Encoding operators.

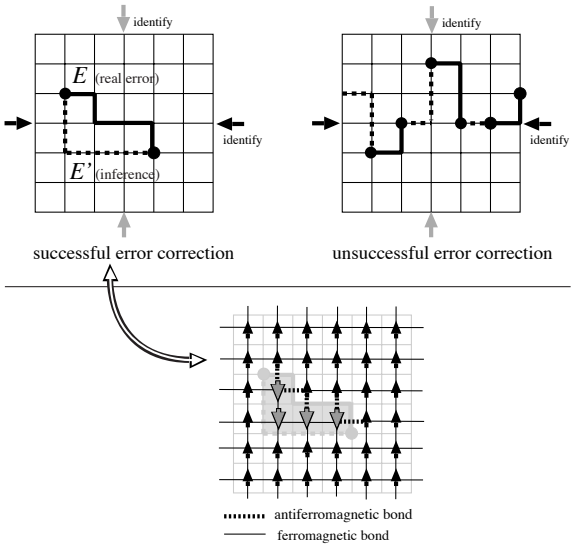

Fig. 3. Examples of the error correction procedure and the mapping between the toric code and the RBIM.

eigenstates of these operators. The eigenvalues of these check operators are ± 1, and therefore the stored quantum state $|\Psi\rangle$ may be chosen as a simultaneous eigenstate of all these check operators with the eigenvalue 1.

Besides the check operators, we can also define the encoding operators which are used for observing stored information. The operators $\bar{X}_1, \bar{X}_2, \bar{Z}_1, \bar{Z}_2$ are defined by

$$\bar{X}_{1,2} = \bigotimes_{l \in C_{X_{1,2}}} \sigma_l^x, \quad \bar{Z}_{1,2} = \bigotimes_{l \in C_{Z_{1,2}}} \sigma_l^z, \tag{3}$$

where the bond sets C_{X_1,X_2,Z_1,Z_2} are depicted in Fig.2. The key property of these operators is their commutativity with all check operators, so the information by these operators is independent of the ones by the check operators. They are the very information to be protected from noises. (Strictly, the pairs $\bar{X}_{1,2}$ and $\bar{Z}_{1,2}$ are not independent of each other.) Note that these operators are defined by the homologically nontrivial sets of bonds, which is crucial for the construction of the error correction procedure. In addition, they are used for the creation and the detection of Z_2 vortices in the Ising gauge theory appearing in other contexts such as cuprate superconductivity.[5)]

Next we move on to the error correction procedure. Let us consider the case that a "chain" of phase errors appears as shown in Fig.3. Since we prepare the original state whose eigenvalues of check operators are all positive (1), we can find the ends of the error chain from the positions of wrong sign (-1) of the check operators X_i, called syndrome. (A bit-flip error chain can be treated similarly using Z_P.) From the ends (black circles, syndrome) of the real error chain (E, solid) we infer the correction chain (E', broken). We show two examples of the correction, a successful (top left) and an unsuccessful (top right) ones. The case on the left, where the two chains form a homologically trivial loop, does not affect the eigenvalues of the homologically nontrivial encoded operators. On the other hand, the chains in the right one form a homologically nontrivial loop, so this procedure affects the stored information. Such an unsuccessful case often occurs when the error rate becomes large because many/long error chains may appear, which suggests the existence of the accuracy threshold for the error rate.

This procedure can be mapped to the random spin model[2)] as depicted at the bottom of Fig.3. By this mapping, a correction pattern can be regarded as a specific configuration of spins and bonds on the *dual* lattice in the $2D$ $(\pm J)$ RBIM. The error rate (denoted by $1-p$) of a single qubit corresponds to the probability of antiferromagnetic interaction for each bond. The ratio of error probability to non-error probability $(1-p)/p$ is identical with the edge Boltzmann factor of local interaction corresponding to the ratio of unfavorable and favorable spin alignments, e^{-K}/e^{K} as seen in Fig.3. Thus the $2D$ RBIM under consideration lies on the Nishimori line (NL), $e^{-2K} = (1-p)/p$.[6)]

The relation between the phase diagram and the accuracy threshold is explained intuitively as follows.[2)] In the magnetically ordered phase of the RBIM, the correction is successful since the islands of reversed spins are not extensively large in the ordered phase as in Fig.3, which implies that the difference between E and E' is not significant. On the other hand, in the disordered phase the error correction fails and encoded information is lost. Thus the critical probability on the NL (or the multicritical point) gives the accuracy threshold.

However the mapping onto the $2D$ RBIM is incomplete because the correction procedure is not always done perfectly and there exists another kind of error due to the imperfect measurement of the check operator eigenvalues. If we take it into account and observe the history of the appearance and the correction of erroneous qubits on the $2D$ lattice, the procedure should be considered on the $3D$ cubic lattice with two spatial dimensions and one time dimension, where the errors by the imperfect measurement behave like the erroneous qubits along the time direction. When we consider the mapping from the error correction procedure for such a system onto a random spin system, the locations of erroneous qubits (bonds) are mapped to the ones of the *dual* plaquettes, where the signs of the interaction are reversed, accompanied by

an appropriate configuration of spins on the dual bonds just like the mapping onto the RBIM. Thus it follows that the corresponding spin system is the $3D$ RPGM,[2] whose Hamiltonian is

$$H = -J \sum_{P} \tau_P \prod_{l \in \partial P} S_l, \tag{4}$$

where P, l are a plaquette and a bond on the lattice, respectively. τ_P is the random variable which takes 1 with probability p and -1 with $1-p$. Therefore we are asked to investigate the location of the multicritical point, or the critical point between the confinement and the Higgs phases on the NL for the $3D$ RPGM. Note that we restrict ourselves to the case that the two rates, the error rate and the imperfect measurement rate, are equal, namely $1-p$.

3. Duality of random spin models

From the discussion of the previous section, we know that our task is to identify the location of the multicritical point for the $3D$ RPGM. We may consider that its location can be obtained by the numerical studies, which turns out not necessarily easy for the RPGM, in particular if we demand high precision. In this article, we will analytically address this problem with the aid of a new tool, that is, the duality.

From now on, let us focus on the dual structure of random spin models. Before treating random models, we briefly review the duality of non-random Z_2 spin systems.[7] As is known, there are some self-dual spin models as listed below.

- 2D Ising model on the square lattice (Kramers-Wannier duality)
- $4D$ Z_2 gauge model on the hypercubic lattice

The Hamiltonian of the Z_2 gauge model is defined by Eq.(4) without randomness τ_P.

In addition, several mutually-dual pairs are also known.

- $2D$ Ising model on the triangular lattice $\leftrightarrow$ $2D$ Ising model on the hexagonal lattice
- $3D$ Ising model on the cubic lattice $\leftrightarrow$ $3D$ Z_2 gauge model on the cubic lattice

Remember that the latter pair is the one which has a relation with the problem in this article.

Now we turn to the duality of random models. The Hamiltonian of the general random models with Z_2 spins and bimodal randomness can be written as,

$$H = -J \sum_{C} \tau_C \prod_{\mathbf{x} \in \partial C} S_{\mathbf{x}}, \tag{5}$$

where C is the r-dimensional element on the lattice and $S_{\mathbf{x}}$ is the Z_2 spin defined on $r-1$ dimensional element. τ_C is the random coupling which takes the value 1 with probability p and -1 with $1-p$. For $r=1$ and $r=2$ the system corresponds to the RBIM and the RPGM, respectively.

As in Ref.3, the replica technique is useful for averaging over random variables. Let us consider the n-replicated system and define the averaged Boltzmann factor x_m for the specified element $\mathbf{x}$. x_m corresponds to the configuration $\prod_{\mathbf{x} \in \partial C} S_x = 1$ in $n-m$ replicas and -1 in m replicas. The explicit form of x_m is (see Ref.3)

$$x_m(p,K) = p e^{(n-2m)K} + (1-p) e^{-(n-2m)K}, \tag{6}$$

where $K = \beta J$. Note that the case of $n=1$ and $p=1$ corresponds to a nonrandom system discussed above.

The averaged n-replicated partition function is a function of these Boltzmann factors,

$$Z_n\{x_0(p,K), x_1(p,K), \ldots, x_n(p,K)\} \equiv [Z^n]_{\mathrm{av}}, \tag{7}$$

where $[\ ]_{\mathrm{av}}$ means random average.

We can also define the dual averaged Boltzmann factor $x^*_m(p,K)$ on the dual lattice by the two-component Fourier transformation.[3,8] They are obtained as

$$\begin{aligned} x^*_{2m}(p,K) &= 2^{\frac{n}{2}} \cosh^{n-2m} K \sinh^{2m} K, \\ x^*_{2m+1}(p,K) &= 2^{\frac{n}{2}} (2p-1) \cosh^{n-2m-1} K \sinh^{2m+1} K, \end{aligned} \tag{8}$$

where m is an integer.

We consider the case that the system is self-dual when we remove randomness such as the $2D$ RBIM and the $4D$ RPGM. Then we can express the self-duality of the n-replicated partition function using x_m and x^*_m,

$$Z_n\{x_0, \ldots, x_n\} = Z_n\{x^*_0, \ldots, x^*_n\}, \tag{9}$$

up to an overall constant. Self-duality is recognized by the fact that Z_n is invariant if we exchange $x_m(p,K) \leftrightarrow x^*_m(p,K)$ for all m simultaneously.

The n-replicated partition function Z_n is a complicated function of the averaged Boltzmann factors, so a conjecture[3] should be made for the investigation of the multicritical point by using all-up Boltzmann factors as

$$x_0(p_c, K_c) = x^*_0(p_c, K_c), \tag{10}$$

where subscript c means criticality. From this equation and the condition of the NL, we can expect to identify the location of the multicritical point for self-dual cases.

However this argument cannot be applied to non-self-dual random systems such as mutually-dual pairs. To make the argument applicable to such cases, let us consider the product of the n-replicated partition functions for the two systems, which are mutually dual if randomness is removed. The product satisfies

$$\begin{aligned} &Z^{\mathrm{O}}_n\{x_0(p^{\mathrm{o}}, K^{\mathrm{o}}), \ldots x_n\} Z^{\mathrm{D}}_n\{x_0(p^{\mathrm{d}}, K^{\mathrm{d}}), \ldots x_n\} \\ =&Z^{\mathrm{O}}_n\{x^*_0(p^{\mathrm{d}}, K^{\mathrm{d}}), \ldots x^*_n\} Z^{\mathrm{D}}_n\{x^*_0(p^{\mathrm{o}}, K^{\mathrm{o}}), \ldots x^*_n\}, \end{aligned} \tag{11}$$

where $p^{\mathrm{o}}, K^{\mathrm{o}}$ and $p^{\mathrm{d}}, K^{\mathrm{d}}$ are parameters of the original and the dual models, respectively. (Here we restrict ourselves to the properties on the NL.) Z^{O}_n and Z^{D}_n mean the replicated partition functions of the original and the dual models. Therefore the product is invariant under the simultaneous exchange $x_m(p^{\mathrm{o}}, K^{\mathrm{o}}) \leftrightarrow x^*_m(p^{\mathrm{d}}, K^{\mathrm{d}})$ and $x_m(p^{\mathrm{d}}, K^{\mathrm{d}}) \leftrightarrow x^*_m(p^{\mathrm{o}}, K^{\mathrm{o}})$ for all m.

Considering this property, it is natural to use the product of the all-up Boltzmann factors for the modification of the conjecture. Then we arrive at the new conjecture by symmetrizaton of Eq.(10) as follows,

$$x_0(p^{\mathrm{o}}_c, K^{\mathrm{o}}_c) x_0(p^{\mathrm{d}}_c, K^{\mathrm{d}}_c) = x^*_0(p^{\mathrm{o}}_c, K^{\mathrm{o}}_c) x^*_0(p^{\mathrm{d}}_c, K^{\mathrm{d}}_c). \tag{12}$$

We expect that the two multicritical points are related by

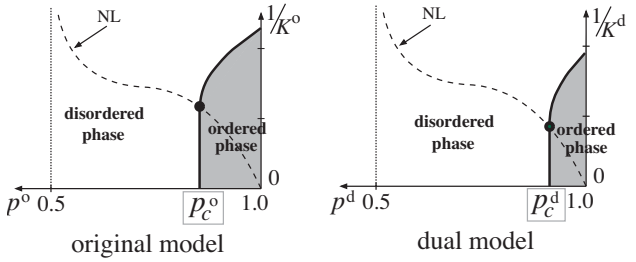

Fig. 4. Schematic phase diagrams of mutually-dual random spin models with bimodal randomness. The relation (13) will be satisfied by the two multicritical points.

this equation. This is the generalization of the conjecture (10) to non-self-dual random systems.

We can check that Eq.(12) (and also Eq.(10)) gives the rigorous relation between two multicritical points in the cases of $n = 1, 2$ and $n \to \infty$ (and the Ising case $r = 1$) using the fact that the replicated systems are equivalent to the well-known nonrandom spin models, which have been studied in detail. More precisely, for $n = 1$ and $n \to \infty$ the system is equivalent to the nonrandom Ising model or for $n = 2$ to the nonrandom 4-state Potts model.

If we take the quenched ($n \to 0$) limit of Eq.(12) combined with the condition of the NL, we obtain the relation for the multicritical points from $O(n^1)$ terms,

$$H(p_c^{\mathrm{o}}) + H(p_c^{\mathrm{d}}) = 1. \tag{13}$$

This is the main result which we would like to stress in this article. Here $H(p)$ is the binary entropy function, $H(p) \equiv (-p \log p - (1-p) \log(1-p))/\log 2$.

We can check the conjecture (13) numerically in several cases. If the system is self-dual such as the $2D$ RBIM and the $4D$ RPGM, p_c must satisfy $H(p_c) = \frac{1}{2}$ which gives $p_c = 0.889972\ldots$,[3,4] and this value is supported by the several numerical studies.[9–11] The relation for non-self-dual cases can be checked as well. We estimated the location of the multicritical points for the $2D$ RBIM on the triangular and the hexagonal lattices by the non-equilibrium relaxation method.[10] For the hexagonal lattice $p_c^{\mathrm{o}} = 0.930(5)$ or $0.347 < H(p_c^{\mathrm{o}}) < 0.384$, and for the triangle $p_c^{\mathrm{d}} = 0.835(5)$ or $0.634 < H(p_c^{\mathrm{d}}) < 0.658$.[12] The results yield $0.981 < H(p_c^{\mathrm{o}}) + H(p_c^{\mathrm{d}}) < 1.042$, which is consistent with our conjecture (13). From the checks above, we can conclude that the conjecture (13) is reliable and is expected to hold for mutually-dual random systems generally.

Finally, we come to the stage to estimate the location of the multicritical point for the $3D$ RPGM using the duality between two $3D$ models. By the same argument as the $2D$ case, it is easily found that the RBIM and the RPGM on the $3D$ cubic lattice are mutually dual. The location of the multicritical point for the $3D$ RBIM is calculated numerically with high precision, $p = 0.7673(3)$.[13] From this result we can estimate the accuracy threshold of the toric code at about 3.46% (which corresponds to the location of the multicritical point for the $3D$ RPGM at about $p = 0.9653$) using the dual relation (13). This result confirms the reliability of an independent numerical result[14] (about 3.3%), and also serves as a verification of our conjecture (13) combined with this numerical result.

4. Conclusion

We overviewed the definition of the toric code and the dual formalism of self-dual random spin models. We extended this dual formalism to the non-self-dual random systems and obtained a conjecture which gives the relation between the locations of two multicritical points of mutually-dual random model pairs. We checked the conjecture in several cases and conclude that the conjecture is (not rigorously proved but) reliable. This conjecture is interesting from the viewpoint of the statistical mechanics and the rigorous proof is desired.

Using the dual structure and the conjecture, we estimated the accuracy threshold of the toric code, which is estimated to be about 3.46% using the extensive numerical result for the $3D$ RBIM and the dual relation (13), which is consistent with the numerical result for the $3D$ RPGM by another group. We therefore conclude that the problem of numerical determination of accuracy threshold of the toric code has been settled.

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