Energy Gap at First-Order Quantum Phase Transitions: An Anomalous Case

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We show that the rate of closing of the energy gap between the ground state and the first excited state, as a function of system size, behaves in many qualitatively different ways at first-order quantum phase transitions of the infinite-range quantum XY model. Examples include polynomial, exponential and even factorially-fast closing of the energy gap, all of which coexist along a single axis of the phase diagram representing the transverse field. This variety emerges depending on whether or not the transverse field assumes a rational number, as well as on how the series of system size is chosen toward the thermodynamic limit. We conclude that there is no generically applicable rule for relating the rate of gap closing to the order of quantum phase transitions as is often implied in many studies, particularly in relation to the computational complexity of quantum annealing in its implementation as quantum adiabatic computation.

KEYWORDS: quantum phase transition, quantum annealing, energy gap

1. Introduction

Quantum annealing is a generic algorithm to solve combinatorial optimization problems, expressed in terms of the Ising model, using quantum fluctuations or quantum tunneling.[1–6] The system, typically the transverse-field Ising model with complex interactions, is initially set to the ground state of a trivial quantum Hamiltonian, e.g., the transverse-field term, and then is driven by the time-dependent Schrödinger equation toward the ground state of the final Hamiltonian corresponding to the solution of a given combinatorial optimization problem. A descendant of quantum annealing is quantum adiabatic computation,[7] in which the system is assumed to follow the instantaneous ground state of a time-dependent quantum Hamiltonian. We study this latter realization of quantum annealing in the present paper.

The initial and final states are quite different from each other, the former being trivial and the latter highly non-trivial. This implies that the initial and final states belong to different thermodynamic phases. Therefore, after the thermodynamic (large-size) limit is taken, the system is expected to undergo a quantum phase transition.

It is often the case that the energy gap between the ground state and the first excited state closes at a quantum phase transition point, which causes difficulties because the adiabatic condition of quantum mechanics states that the time scale to stay in the instantaneous ground state is inversely proportional to the square of the energy gap.

One of the main interests in combinatorial optimization problems is the computational complexity, i.e., the time necessary for a given algorithm to reach the solution as a function of the system size (problem size). The system size is usually large but finite, and thus our task amounts to the determination of whether the gap closes very quickly, typically exponentially, as a function of system size or relatively slowly or polynomially. It is generally believed, and is indeed the case in many instances, that the gap closes exponentially fast at first-order quantum phase transitions whereas it is polynomial for second-order transitions. Many researchers have therefore been attempting to determine the order of transitions in quantum systems representing quantum annealing. See Refs.[8–11] and references cited therein.

An interesting counterexample was presented by Cabrera and Jullien[12] who showed that the first-order quantum phase transition in the one-dimensional transverse-field Ising model accompanies a polynomial closing of the energy gap if one imposes an antiperiodic boundary condition. See also Ref.[13] for essentially the same result. In the present paper, we give another quite unusual example where the energy gap closes in widely different ways – polynomial, exponential, and factorial – depending strongly on the value of the parameter in the Hamiltonian as well as on the choice of the sequence of system size toward the thermodynamic limit.

In the next section, the model system is described and its thermodynamic behavior is analyzed. §3 constitutes the main body of this paper, where the behavior of the energy gap for finite-size systems is studied in detail. The final section is devoted to discussions.

2. Model and Its Behavior in the Thermodynamic Limit

We study the infinite-range XY model in transverse and longitudinal fields,

$$H = -\frac{1}{N}\Big[\Big(\sum_{i=1}^{N} S_{x,i}\Big)^2 + \Big(\sum_{i=1}^{N} S_{y,i}\Big)^2\Big] - \Gamma \sum_{i=1}^{N} S_{z,i} - h \sum_{i=1}^{N} S_{x,i}, \tag{1}$$

where N is the system size and $S_{\alpha,i}$ ($\alpha = x, y, z$) is the αth component of a spin-1/2 operator at site i. In the absence of longitudinal field ($h = 0$), this system is sometimes called the Lipkin-Meshkov-Glick model, first studied in the context of nuclear physics.[14] Since we are

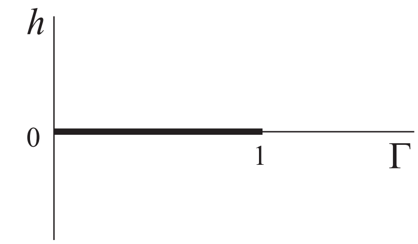

Fig. 1. Phase diagram in the thermodynamic limit. The segment $(0 \le \Gamma < 1, h = 0)$ as drawn bold represents a line of first-order transitions between the phases with $h > 0$ and $h < 0$ and is delimited by a critical point at $(\Gamma = 1, h = 0)$.

often interested in the case with finite longitudinal field ($h \neq 0$), we use a more generic denomination of the infinite-range (quantum) *XY* model. In the absence of longitudinal field, the model has been studied in detail by Botet and Jullien[15] and Dusuel and Vidal.[16] We closely follow these references in this and the next sections. Before embarking on the study of the energy gap for finite-size systems in the next section, we focus our attention on the properties in the thermodynamic limit in this section.

The phase diagram in the ground state can be drawn using the fact that quantum spins appear in the Hamiltonian (1) only as summations over all sites. We therefore use the total spin $S_\alpha = \sum_i S_{\alpha,i}$ $(\alpha = x, y, z)$ to rewrite the Hamiltonian (1) as

$$H = -\frac{1}{N}[(S_x)^2 + (S_y)^2] - \Gamma S_z - hS_x. \tag{2}$$

The ground state of this Hamiltonian belongs to the subspace of the largest total spin, $S = N/2$. For a sufficiently large N, the total spin operator $\boldsymbol{S} = (S_x, S_y, S_z)$ thus behaves (semi-) classically, and we may regard $\boldsymbol{S}$ as a classical vector of length $N/2$:

$$S_x = \frac{1}{2}N\sin\theta\cos\phi, \quad S_y = \frac{1}{2}N\sin\theta\sin\phi, \quad S_z = \frac{1}{2}N\cos\theta. \tag{3}$$

Then the ground state for a given set of values of Γ and h is determined by inspection of the direction of $\boldsymbol{S}$ that gives the lowest value of the energy,

$$\epsilon_g \equiv \frac{H}{N} = -\frac{1}{4}\sin^2\theta - \frac{1}{2}h\sin\theta\cos\phi - \frac{1}{2}\Gamma\cos\theta. \tag{4}$$

The resulting phase diagram in the Γ-h plane is drawn in Fig. 1. A line of first-order transitions exists for $(0 \le \Gamma < 1, h = 0)$, which is delimited by a critical point at $(\Gamma = 1, h = 0)$. The magnetization in the x direction jumps from a positive value to a negative value as h crosses 0 from $h > 0$ to $h < 0$ for $0 \le \Gamma < 1$. The line segment $(0 \le \Gamma < 1, h = 0)$ hence represents a set of first-order transitions. We study the properties of these first-order transitions in this work.

Quantum corrections to the above-mentioned classical limit yield the energy gap $\Delta(\Gamma, h)$ between the ground state and the first excited state.[16,17] Note here that the following method of using the Holstein-Primakoff transformation gives the energy gap in the thermodynamic limit. The rate of gap closing for finite-size systems should be discussed using other approaches, as described in the next section.

Let us first rotate the axes such that the vector $\boldsymbol{S}$ lies along the new z axis,

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix} \begin{pmatrix} \tilde{S}_x \\ \tilde{S}_y \\ \tilde{S}_z, \end{pmatrix}, \tag{5}$$

where θ_0 is the value of θ for minimizing Eq. (4) and is a function of h and Γ. The other angle ϕ is clearly 0 in the ground state. We apply the Holstein-Primakoff transformation and its semi-classical approximation,

$$\tilde{S}_z = \frac{1}{2}N - a^\dagger a, \tag{6}$$

$$\tilde{S}_+ = \sqrt{N - a^\dagger a}\, a \approx \sqrt{N}\, a, \tag{7}$$

$$\tilde{S}_- = a^\dagger \sqrt{N - a^\dagger a} \approx \sqrt{N}\, a^\dagger. \tag{8}$$

The Hamiltonian (2) now becomes a quadratic form of a and $a^\dagger$, up to an additive constant,

$$H = N\epsilon_g + \frac{\sin^2\theta_0}{4}\left[a^2 + (a^\dagger)^2\right] + \left(\frac{3}{2}\sin^2\theta_0 - 1 + h\sin\theta_0 + \Gamma\cos\theta_0\right) a^\dagger a. \tag{9}$$

A Bogoliubov transformation diagonalizes this expression into[9]

$$H = N\epsilon_g + \Delta(\Gamma, h)\, b^\dagger b. \tag{10}$$

Here, $b^\dagger$ and b are Bogoliubov boson operators and $\Delta(\Gamma, h)$ is the energy gap given by

$$\Delta(\Gamma, h) = \sqrt{\left(\frac{3}{2}\sin^2\theta_0 - 1 + h\sin\theta_0 + \Gamma\cos\theta_0\right)^2 - \frac{1}{4}\sin^4\theta_0}, \tag{11}$$

where θ_0, as defined above, is a function of Γ and h. Equation (11) gives the energy gap in the thermodynamic limit as a function of Γ and h and is drawn as a function of h in Fig. 2 for $\Gamma = 0$. It is straightforward to verify that this gap continuously approaches 0 as $|h| \to 0$ for $0 \le \Gamma < 1$. This continuous approach to 0 as $|h| \to 0$ is reminiscent of a second-order transition since a first-order transition usually involves a jump of the gap in the thermodynamic limit as exemplified in Fig. 2 of Ref.[11] The gap is 0 exactly at the transition point and remains finite in the neighborhood. The present continuous behavior at a first-order transition may be related to the continuous nature of the broken symmetry.

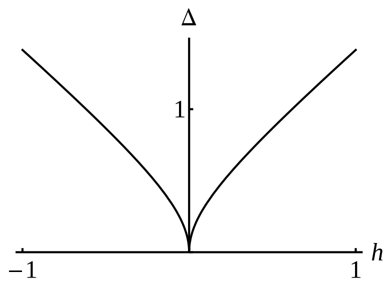

Fig. 2. Energy gap in the thermodynamic limit as a function of h for $\Gamma = 0$.

3. Energy Gap as a Function of the System Size

We next investigate the system-size dependence of the energy gap at the transition point. The size N is finite in this section. Our interest is in the first-order transition across $h = 0$ for $0 \le \Gamma < 1$. We therefore set $h = 0$ and carefully trace the analysis by Dusuel and Vidal[16) to identify insufficiencies therein.

When $h = 0$, the Hamiltonian (2) is a function of the total spin, $\boldsymbol{S}^2 = S(S+1) = \frac{1}{2}N(\frac{1}{2}N + 1)$, and its z component $S_z = M$:

$$\begin{aligned} H &= -\tfrac{1}{N}(\boldsymbol{S}^2 - S_z^2) - \Gamma S_z = -\tfrac{1}{N}\left(\tfrac{1}{2}N(\tfrac{1}{2}N+1) - M^2\right) - \Gamma M \\ &= -\tfrac{1}{2}(\tfrac{1}{2}N + 1) + \tfrac{M^2}{N} - \Gamma M \equiv E_g(N, M). \end{aligned} \tag{12}$$

The ground state has M that minimizes this expression. A simple differentiation gives

$$M = \frac{\Gamma N}{2}. \tag{13}$$

In general, this value of M cannot be realized because M assumes only integers when N is even and half-integers (half of odd numbers) for odd N in the range $-\frac{1}{2}N \le M \le \frac{1}{2}N$. This latter condition on the range is satisfied when $0 \le \Gamma < 1$. To examine the condition of integer M for even N, we divide $\Gamma N/2$ into integer and fractional parts,

$$\frac{\Gamma N}{2} = \left\lfloor \frac{\Gamma N}{2} \right\rfloor + \delta \quad (0 \le \delta < 1). \tag{14}$$

Then the value of M to minimize (12), to be written as M_0, is the integer closest to $\Gamma N/2$:

$$M_0 = \begin{cases} \left\lfloor \dfrac{\Gamma N}{2} \right\rfloor & 0 \le \delta < \frac{1}{2} \\ \left\lfloor \dfrac{\Gamma N}{2} \right\rfloor + 1 & \frac{1}{2} < \delta < 1. \end{cases} \tag{15}$$

The value of M for the first excited state, to be denoted as M_1, is the integer closest to M_0,

$$M_1 = \begin{cases} M_0 + 1 & 0 \le \delta < \frac{1}{2} \\ M_0 - 1 & \frac{1}{2} < \delta < 1. \end{cases} \tag{16}$$

The case $\delta = 1/2$ may occur accidentally for very special values of Γ and N, but does not play insightful roles in the analysis of the generic behavior of the gap and will not be considered here.

We are now ready to evaluate the energy gap for finite-size systems,[18,19]

$$\Delta_N(\Gamma, 0) = E_g(N, M_1) - E_g(N, M_0) = \frac{M_1^2 - M_0^2}{N} - \Gamma(M_1 - M_0). \tag{17}$$

When $0 \le \delta < 1/2$, the insertion of Eqs. (15) and (16) yields

$$\Delta_N(\Gamma, 0) = \frac{2M_0 + 1}{N} - \Gamma = \frac{2}{N}\left(\left\lfloor \frac{\Gamma N}{2} \right\rfloor + \frac{1}{2} - \left\lfloor \frac{\Gamma N}{2} \right\rfloor - \delta\right) = \frac{1 - 2\delta}{N}. \tag{18}$$

Similarly, for $1/2 < \delta < 1$,

$$\Delta_N(\Gamma, 0) = \frac{-2M_0 + 1}{N} + \Gamma = \frac{2\delta - 1}{N}. \tag{19}$$

When N is odd, we divide $\Gamma N/2$ into half-integer and fractional parts,

$$\frac{\Gamma N}{2} = \left\lfloor \frac{\Gamma N}{2} \right\rfloor_{\mathrm{half}} + \delta' \quad (0 \le \delta < 1), \tag{20}$$

where the notation $\lfloor \alpha \rfloor_{\mathrm{half}}$ is the maximum half-integer that is smaller than or equal to α. Then the same expression for energy gap (19) can be derived with δ replaced by δ'.

These results (18) and (19), first derived in Ref.,[16] may superficially be regarded as evidence of a polynomially-closing gap. We argue that this is not necessarily true since δ depends on N and can approach $1/2$ very rapidly as N increases, which implies, from Eqs. (18) and (19), that the gap may behave in unusual ways.

It is useful to consider rational and irrational values of Γ separately. When Γ is a rational number, the values of δ defined in Eq. (14) are restricted to a finite set. Then, $1 - 2\delta$ or $2\delta - 1$ in the numerator of Eq. (18) or Eq. (19) does not approach 0 asymptotically taking infinitely many values as N is increased.[20] In this case, the gap $\Delta_N(\Gamma, 0)$ closes in proportion to N^{-1}. This behavior is already anomalous because the first-order transition across $h = 0$ is accompanied by a polynomial closing of the energy gap.

A more prominent anomaly in the gap is exemplified for irrational Γ by using the following sequence:

$$a_{n+1} = 2^{a_n} \quad (n = 0, 1, 2, \cdots), \quad a_0 = 1. \tag{21}$$

Let us correspondingly set the value of Γ to

$$\Gamma = \sum_{n=1}^{\infty} \frac{1}{a_n}. \tag{22}$$

It is easy to show that this Γ lies in the range $1/2 < \Gamma < 1$ (see Appendix). The sequence of system size is specified as

$$N_n = a_n. \tag{23}$$

Then the corresponding δ, denoted as $\delta(N_n, \Gamma)$, satisfies

$$\delta(N_n, \Gamma) = \frac{1}{2} + \frac{N_n 2^{-N_n}}{2} \sum_{k=n+1}^{\infty} \frac{a_{n+1}}{a_k} \tag{24}$$

as shown in Appendix. Since the series appearing at the end of the above equation converges to 1 as $n \to \infty$ (see Appendix), we find

$$\delta(N_n, \Gamma) = \frac{1}{2} + \mathcal{O}(N_n 2^{-N_n}). \tag{25}$$

According to Eq. (19), this result (25) means that the gap closes exponentially.[21)]

If we choose another sequence of system size using the same series $\{a_n\}$ as

$$N_n = 2a_n, \tag{26}$$

and use the same Γ as in Eq. (22), an analysis similar to that above reveals

$$\delta(N_n, \Gamma) = \frac{N_n 2^{-N_n/2}}{2} \sum_{k=n+1}^{\infty} \frac{a_{n+1}}{a_k}. \tag{27}$$

This δ vanishes as $n \to \infty$ and hence the gap (18) closes polynomially.

We have therefore established that, for the same irrational Γ, the rate of closing of the energy gap behaves significantly differently depending on the choice of the sequence of system size toward the thermodynamic limit. This is good news for quantum annealing since we can avoid an exponential computational complexity (exponential gap closing) merely by choosing the right sequence.[22)]

It is also possible to let the gap close enormously quickly, inversely proportionally to the factorial of system size. Consider the following sequence

$$a_{n+1} = a_n! \quad (n = 1, 2, 3, \cdots), \quad a_1 = 3, \quad N_n = a_n, \tag{28}$$

and choose Γ as in Eq. (22). Then, as described in Appendix,

$$\delta(N_n, \Gamma) = \frac{1}{2} + \frac{a_n}{2a_{n+1}} \sum_{k=n+1}^{\infty} \frac{a_{n+1}}{a_k} = \frac{1}{2} + \frac{a_n}{2a_n!} \cdot \mathcal{O}(1). \tag{29}$$

We therefore find that the gap closes as

$$\Delta_{N_n}(\Gamma,0)=\frac{2\delta(N_n,\Gamma)-1}{N_n}=\frac{1}{a_n}\frac{a_n}{a_n!}\cdot\mathcal{O}(1)=\frac{1}{a_n!}\cdot\mathcal{O}(1)=\mathcal{O}\left(\frac{1}{N_n!}\right). \tag{30}$$

As discussed in Appendix, the values of Γ giving these anomalous behaviors of the gap exist densely on the real-Γ axis with the same cardinal number as that of real numbers. We conclude that drastically different rates of gap closing can arise by any infinitesimal change of Γ and a meticulous choice of the size sequence.

4. Conclusions

We have shown that the first-order phase transition in the infinite-range quantum XY model has a continuously vanishing energy gap in the thermodynamic limit as a function of the longitudinal field h when it crosses the transition point at $h=0$. This continuous change of the energy gap across the transition point is in marked contrast to the discontinuous behavior in the transverse-field Ising model.

Our main discovery in the present work is that the energy gap at a first-order transition point behaves in widely different ways as a function of system size toward the thermodynamic limit. Polynomial, exponential, and factorial rates of gap closing have been found to exist, depending strongly on the rationality of the parameter Γ and the sequence of system size. This is quite an unexpected and astonishing result. Not only is the order of quantum phase transitions unrelated to the rate of gap closing, but it has also been revealed that completely different rates coexist along the Γ axis in the same manner as rational and irrational numbers coexist along the axis of real numbers. Such anomalous properties may be specific to the present system, but it is certainly worth further studies to verify this point.

Appendix

In this Appendix, we derive several properties of $\{a_n\}$, Γ, and $\Delta(N_n,\Gamma)$, as referred to in §3.

First, it is easy to prove $1/2<\Gamma<1$. Since $a_1=2, a_2=2^2, a_3=2^{2^2}, a_4=2^{2^{2^2}}, a_5=2^{2^{2^{2^2}}}, \cdots$, the inequality $1/2<\Gamma$ trivially holds. The other inequality $\Gamma<1$ is shown as

$$\Gamma=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^{2^2}}+\cdots<\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots=1. \tag{A·1}$$

We next derive Eq. (24). By the definitions (22) and (23),

$$\frac{\Gamma N_n}{2}=\frac{a_n}{2}\sum_{k=1}^{\infty}\frac{1}{a_k}=\frac{1}{2}\left(\sum_{k=1}^{n-1}\frac{a_n}{a_k}+1+\sum_{k=n+1}^{\infty}\frac{a_n}{a_k}\right). \tag{A·2}$$

The first summation on the right-hand side for k up to $n-1$ is a multiple of 2 according to the definition (21) of $\{a_n\}$, and thus half of it, the first term on the right-hand side of the above equation, is an integer. Hence,

$$\delta(N_n, \Gamma) = \frac{\Gamma N_n}{2} - \left\lfloor \frac{\Gamma N_n}{2} \right\rfloor = \frac{1}{2} + \frac{a_n}{2a_{n+1}} \sum_{k=n+1}^{\infty} \frac{a_{n+1}}{a_k}. \tag{A·3}$$

Using Eqs. (21) and (23) we obtain Eq. (24).

The series on the right-hand side of Eq. (A·3) converges to 1 as $n \to \infty$. To see it,

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{a_{n+1}}{a_k} &= 1 + \frac{a_{n+1}}{a_{n+2}} + \frac{a_{n+1}}{a_{n+3}} + \cdots \\ &= 1 + \frac{a_{n+1}}{a_{n+2}} \left(1 + \frac{a_{n+2}}{a_{n+3}} + \frac{a_{n+2}}{a_{n+4}} + \cdots \right) = 1 + \frac{a_{n+1}}{a_{n+2}}\, c\,, \end{aligned} \tag{A·4}$$

where the series in parentheses clearly converges very rapidly and thus c is a bounded constant. Since $a_{n+2} = 2^{a_{n+1}}$ and $a_{n+1} \to \infty$ as $n \to \infty$, the second term in the final expression of the above equation converges to 0 in the limit $n \to \infty$.

For a different sequence (28), Eq. (29) can be derived in a similar manner as above. Indeed, the middle expression of Eq. (29) is nothing but Eq. (A·3).

Let us next prove that the cardinal number of the set of values of Γ that satisfy the properties described in the text is the same as the cardinal number of real numbers, $\aleph$. Let us select an arbitrary real number $x \in (0, 1)$ and write it in decimal form,

$$x = \sum_{n=0}^{\infty} b_n\, 10^{-n}, \tag{A·5}$$

and define the following Γ corresponding to this x:

$$\Gamma = \sum_{n=0}^{\infty} \frac{2b_n + 1}{a_{n+1}}, \tag{A·6}$$

where a_n is sequence (21). The same analysis as in §3 can be developed for this Γ. In particular, this Γ is irrational and $\Delta(N_n, \Gamma)$ can be chosen to decay exponentially. Since the mapping from x to Γ is an injection, the cardinal number of x is equal to or smaller than that of Γ. On the other hand, since Γ is a real number, its cardinal number cannot exceed that of x. This completes the proof that x and Γ share the same cardinal number.

Finally, we prove that Γ satisfying the properties given in §3 exists densely in the range $(0, 1)$. This is to show that, for any $a, b \in (0, 1)$ $(b > a)$, there exists a Γ between them, $a < \Gamma < b$. For this purpose, we first note that there exists $k \in \mathbb{N}$ such that

$$b - a > \frac{1}{2^k}. \tag{A·7}$$

For such k, there also exists $N \in \mathbb{N}$ such that

$$a < \frac{N}{2^k} < b. \tag{A·8}$$

Then, for sufficiently large k with the above property, we have

$$\max\left(b - \frac{N}{2^k}, \frac{N}{2^k} - a\right) > \frac{1}{2^{k+1}}. \tag{A·9}$$

By the way, for the sequence (21), there exists n satisfying

$$a_n > 2^{k+2}. \tag{A·10}$$

Then, the following Γ has the properties discussed in §3 and satisfies $\Gamma \in (a, b)$,

$$\Gamma = \begin{cases} \dfrac{N}{2^k} + \displaystyle\sum_{j=n}^{\infty} \frac{1}{a_j} & \text{if} \quad b - \dfrac{N}{2^k} > \dfrac{N}{2^k} - a \\ \dfrac{N}{2^k} - \displaystyle\sum_{j=n}^{\infty} \frac{1}{a_j} & \text{if} \quad b - \dfrac{N}{2^k} < \dfrac{N}{2^k} - a \end{cases} \tag{A·11}$$

because the series appearing above is bounded by 2^{-k-1},

$$\sum_{j=n}^{\infty} \frac{1}{a_j} < \frac{1}{a_n} + \frac{1}{2a_n} + \frac{1}{2^2 a_n} + \frac{1}{2^3 a_n} + \cdots = \frac{2}{a_n} < \frac{1}{2^{k+1}} \tag{A·12}$$

and the argument to derive the anomalous properties given in §3 can be applied to this Γ almost as is.

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18) Note that the gap $\Delta(\Gamma, h)$ discussed in the previous section is the limit of $\Delta_N(\Gamma, h)$ as $N \to \infty$.

19) In the strict absence of a longitudinal field $h = 0$, the even-odd parity of M is conserved and thus the proper gap is the one between the ground state and the second excited state. However, since we are interested in quantum annealing driven by the longitudinal field, which changes the parity of M, it is legitimate to study the gap defined here.

20) In some cases, δ may remain exactly $1/2$ for a carefully chosen sequence of N. However, this does not represent the asymptotic behavior we are interested in.

21) Note that Γ defined by Eq. (22) is irrational because the corresponding δ in Eq. (24) assumes infinitely many values as a function of N_n.

22) It is at our disposal to choose an appropriate sequence of system size in the realization of quantum annealing, numerical or experimental.