

Mean-field theory is exact for the random-field model with long-range interactions

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We study the random-field Ising model with long-range interactions and show the exactness of the mean-field theory under certain mild conditions. This is a generalization of the result of Mori for the non-random and spin-glass cases. To treat random fields, we evoke the self-averaging property of a function of random fields, without recourse to the replica method. The result is that the mean-field theory gives the exact expression of the canonical free energy for systems with power-decaying interactions if the power is smaller than or equal to the spatial dimension.

KEYWORDS: long-range interacting system, mean-field theory, random fields

1. Introduction

Long-range interacting systems attract attention in recent years since such systems have some peculiar properties, for example, negative specific heat in the microcanonical ensemble and ensemble inequivalence.¹⁻⁷ The latter property is that the physical properties differ between microcanonical ensemble and canonical ensemble. A typical example of long-range interaction is the power-law potential which decays as $1/r^\alpha$, where r is the distance between particles/sites. When the effective range of interaction is almost the same as the system size, the system is non-additive: The whole system cannot be divided into subsystems under the condition that the energy of the whole system is equal to the sum of the energies of subsystems, resulting in the peculiar properties. In astrophysics, long-range interacting systems have been investigated by many researchers, for example, for self-gravitating systems.^{1,2,8}

An important property of some of the long-range interacting systems is the exactness of the mean-field theory, which is defined that the free energy is exactly equal to that of the

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corresponding mean-field system. Many studies on specific examples suggest that long-range interacting systems have this property.^{9–14} This property has been proved to hold for a class of generic non-random spin systems and spin-glasses by Mori.^{15–17}

However, there has been no report so far on the exactness of the mean-field theory of long-range interacting spin systems in random fields.¹⁸ Random fields give each site characteristics different from other sites whereas long-range interactions tend to erase strong site dependence due to their averaging properties over many sites. This implies that long-range interactions and random fields have conflicting effects, which may be worth detailed studies. It should also be noticed that the corresponding mean-field model has been reported to have peculiar properties.¹⁹ Another point to be remarked is that Mori used the replica method with integer replica number to discuss the spin-glass case, which makes his analysis incomplete. It is thus worthwhile to further study the effects of randomness in long-range interacting systems. These observations motivate us to investigate the present system.

Our basic strategy is to generalize the method of Mori to accommodate random fields without recourse to the replica method. A long-range interacting system in random fields is introduced in Section 2. In Section 3 we calculate the free energy and then prove the exactness of the mean-field theory for systems without conservation of magnetization. Also, conditions are given for the exactness of the mean-field theory for systems with conserved magnetization. Section 4 summarizes this paper.

2. Model

In this section, a random-field spin model with long-range interactions is introduced. The system size is $N = L^d$, where L is the linear size and d is the spatial dimension. Periodic boundary conditions are imposed. The Hamiltonian is defined as

$$\mathcal{H} = -\frac{J}{2} \sum_{i,j=1,i \neq j}^N K(\mathbf{r}_{i,j}) S_i S_j - \sum_{i=1}^N h_i S_i \quad (J > 0), \quad (1)$$

where S_i is a general classical spin variable at site i with a bounded value, for example, an Ising spin $S_i = \pm 1$, $K(\mathbf{r})$ represents the interaction potential whose range is long in the sense as defined below, and h_i denotes the random field at site i . The variable $\mathbf{r}_{i,j}$ is for the relative position of site i and site j . We do not specify the distribution function of h_i .

This paper deals with the potential as introduced by Mori^{15–17} defined through a non-

negative function defined in \mathbb{R} satisfying $\phi(\mathbf{r}) = \phi(-\mathbf{r})$,

$$K(\mathbf{r}) = \gamma^d \phi(\gamma \mathbf{r}) \geq 0, \quad (2)$$

where γ is a positive number. The function $\phi(\mathbf{r})$ is supposed to satisfy the following conditions,

$$|\phi(\mathbf{r})| < \psi(r), \quad |\nabla \phi(\mathbf{r})| < \frac{d}{dr} \psi(r) \quad (\forall \mathbf{r}, |\mathbf{r}| = r), \quad (3)$$

where $\psi(r)$ is twice-differentiable, convex, integrable, and defined in $(0, \infty)$.

The parameter γ corresponds to the inverse of the interaction range because \mathbf{r} appears as the combination $\gamma \mathbf{r}$ in eq. (2). To consider long-range potentials, we take the limit $\gamma \rightarrow 0$ in two ways: (i) The non-additive limit, $\gamma \rightarrow 0$ with $\gamma L = 1$, and (ii) the van der Waals limit, $L \rightarrow \infty$ first and then $\gamma \rightarrow 0$. In the non-additive limit, the range of interaction is comparable with the linear size of the system, $\gamma^{-1} \propto L$, and the system is non-additive. A typical example is the power-law potential $K(\mathbf{r}) \propto 1/r^\alpha$ ($0 \leq \alpha < d$), in which case $\phi(\mathbf{r}) \propto 1/r^\alpha$. In the van der Waals limit, the interaction range is long but is much smaller than the system size, and the system is additive

The potential $K(\mathbf{r})$ is supposed to be normalized,

$$\sum_{i=1}^N K(\mathbf{r}_i) = 1. \quad (4)$$

In this paper, the exactness of the mean-field theory means that the free energy of the model (1) is exactly equal to that of the mean-field model (infinite-range model),

$$\mathcal{H}_{\text{MF}} = -\frac{J}{2N} \sum_{i,j=1}^N S_i S_j - \sum_i h_i S_i. \quad (5)$$

3. Exactness of the mean-field theory

We now prove the exactness of the mean-field theory.

3.1 Variational expression of the free energy

This section first introduces coarse-grained variables to replace the microscopic spin variables. The whole system is divided into many subsystems $\{B_p\}_{p=1}^\Omega$, each of which is of size l^d , and $\Omega = (L/l)^d$ is the number of subsystems. The linear length l of a subsystem is much larger than the lattice spacing, which is taken to be unity for simplicity, and is much smaller

than L , $1 \ll l$

L . The center site \mathbf{r}_p of a subsystem B_p is chosen to represent the location of the subsystem.

Let us define a coarse-grained variable m_p of subsystem B_p as

$$m_p = \frac{1}{l^d} \sum_{i \in B_p} S_i. \quad (6)$$

We take the continuum limit, $L \rightarrow \infty$ and $l \rightarrow \infty$ with $L/l \rightarrow \infty$. Under this limit, the location \mathbf{x} ($\in \mathbb{I}^d = [0, 1]^d$) in B_p is defined as $\mathbf{x} = \mathbf{r}_p/L$ and the coarse-grained variable as $m(\mathbf{x}) = m_p$. The interaction potential should be normalized under the continuum limit, and hence it is defined as

$$U(\mathbf{x}) = \lim_{L \rightarrow \infty} L^d K(L\mathbf{x}), \quad (7)$$

with the normalization of the potential (4) being modified as

$$\int_{\mathbb{I}^d} d^d x U(\mathbf{x}) = 1, \quad (8)$$

where \mathbb{I} means the interval from 0 to 1 ($\mathbb{I} = [0, 1]$). The free energy per spin for fixed magnetization m is expressed in terms of the coarse-grained variables as long as we consider a potential in the van der Waals limit or a power-law potential $r^{-\alpha}$ with $0 \leq \alpha < d$, as described in Appendix A. The result is

$$f(\beta, m) = \lim_{L, l, L/l \rightarrow \infty} \left(-\frac{1}{\beta L^d} \ln Z(\beta, m) \right) = \inf_{m(\mathbf{x})} \left\{ \mathcal{F}\{\beta; m(\mathbf{x})\}, \int_{\mathbb{I}^d} d^d x m(\mathbf{x}) = m \right\}, \quad (9)$$

where β is the inverse temperature. The generalized free energy per spin $\mathcal{F}\{\beta; m(\mathbf{x})\}$ is expressed as

$$\begin{aligned} \mathcal{F}\{\beta; m(\mathbf{x})\} = & -\frac{J}{2} \iint_{\mathbb{I}^d \otimes \mathbb{I}^d} d^d x d^d y U(\mathbf{x} - \mathbf{y}) m(\mathbf{x}) m(\mathbf{y}) \\ & + J \int_{\mathbb{I}^d} d^d x \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} m(\mathbf{x}) - \frac{1}{\beta} \int_{\mathbb{I}^d} d^d x \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} + h)\} \right], \end{aligned} \quad (10)$$

where $[\dots]$ represents the configurational average over the distribution of random fields $\{h_i\}$, and the function $\mathcal{T}_\beta^{-1}\{\cdot\}$ is the inverse of $\mathcal{T}_\beta\{\cdot\}$ defined as

$$\frac{1}{\beta} \frac{\partial}{\partial s} [\ln Z_0\{\beta(Js + h)\}] = \mathcal{T}_\beta\{s\}. \quad (11)$$

The quantity Z_0 is the trace of the exponential e^{tS} over the single spin variable S ,

$$Z_0(t) = \text{Tr} e^{tS}. \quad (12)$$

Similarly, the free energy per spin of the mean-field model is given as

$$f_{\text{MF}}\{\beta, m\} = -\frac{J}{2}m^2 + J\mathcal{T}_\beta^{-1}\{m\}m - \frac{1}{\beta} \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m\} + h)\} \right]. \quad (13)$$

3.2 Bounds for the free energy

In this section, we derive inequalities on the free energy for fixed magnetization. Using the derived inequality, we prove the exactness of the mean-field theory in the next section.

Since the mean-field free energy (13) is equal to the generalized free energy (9) with $m(\mathbf{x}) = m$ ($\forall \mathbf{x}$), the following inequality holds trivially,

$$f(\beta, m) \leq \mathcal{F}\{\beta; m(\mathbf{x}) = m\} = f_{\text{MF}}\{\beta, m\}. \quad (14)$$

We next derive a lower bound for the generalized free energy $\mathcal{F}\{\beta, m(\mathbf{x})\}$. If we define the Fourier coefficient of the potential as

$$U_n = \int_{\mathbb{I}^d} d^d x U(\mathbf{x}) \cos(2\pi \mathbf{n} \cdot \mathbf{x}) \quad (\forall \mathbf{n} \in \mathbb{Z}^d), \quad (15)$$

the following inequality holds

$$U_n \leq \int_{\mathbb{I}^d} d^d x U(\mathbf{x}) = 1. \quad (16)$$

Under periodic boundary conditions, the local magnetization can also be expanded into a Fourier series

$$m_n = \int_{\mathbb{I}^d} d^d x m(\mathbf{x}) \exp(2\pi i \mathbf{n} \cdot \mathbf{x}). \quad (17)$$

With the Fourier expression, the first term in the generalized free energy (10) is rewritten as

$$\iint_{\mathbb{I}^d \otimes \mathbb{I}^d} d^d x d^d y U(\mathbf{x} - \mathbf{y}) m(\mathbf{x}) m(\mathbf{y}) = \sum_n U_n |m_n|^2. \quad (18)$$

In the same way, we can obtain the following expression to be used later,

$$\int_{\mathbb{I}^d} d^d x m(\mathbf{x})^2 = \sum_n |m_n|^2. \quad (19)$$

Let us denote the largest coefficient of U_n with $\mathbf{n} \neq \mathbf{0}$ as $U_{\max} (\neq 0)$. The first term in the generalized free energy (10) can then be upper bounded as

$$\begin{aligned} \iint_{\mathbb{I}^d \otimes \mathbb{I}^d} d^d x d^d y U(\mathbf{x} - \mathbf{y}) m(\mathbf{x}) m(\mathbf{y}) &\leq m^2 + U_{\max} \sum_{\mathbf{n} \neq \mathbf{0}} |m_n|^2 \\ &= m^2 + U_{\max} \left\{ \int_{\mathbb{I}^d} d^d x m(\mathbf{x})^2 - m^2 \right\}, \end{aligned} \quad (20)$$

where m stands for m_0 . With this inequality, as shown in Appendix B, the generalized free energy is lower-bounded as

$$\mathcal{F}\{\beta; m(\mathbf{x})\} \geq f_{\text{MF}}\{\beta, m\} + U_{\text{max}} \left\{ \int_{\mathbb{I}^d} d^d x f_{\text{MF}}^*\{\beta^*, m(\mathbf{x})\} - f_{\text{MF}}^*\{\beta^*, m\} \right\}, \quad (21)$$

where the function $f_{\text{MF}}^*(\beta^*, m)$ is defined as

$$\begin{aligned} f_{\text{MF}}^*\{\beta^*, m\} &= -\frac{J}{2}m^2 + J \frac{\mathcal{T}_{\beta}^{-1}\{m\}}{U_{\text{max}}}m - \frac{1}{\beta U_{\text{max}}} \left[\ln Z_0 \left\{ \beta U_{\text{max}} \left(J \frac{\mathcal{T}_{\beta}^{-1}\{m\}}{U_{\text{max}}} + \frac{h}{U_{\text{max}}} \right) \right\} \right] \\ &= -\frac{J}{2}m^2 + J(\mathcal{T}_{\beta^*}^*)^{-1}\{m\}m - \frac{1}{\beta^*} \left[\ln Z_0 \left\{ \beta^* \left(J(\mathcal{T}_{\beta^*}^*)^{-1}\{m\} + \frac{h}{U_{\text{max}}} \right) \right\} \right] \end{aligned} \quad (22)$$

with $\beta^* = \beta U_{\text{max}}$. We have defined the function $\mathcal{T}_{\beta^*}^*\{\cdot\}$ as

$$\mathcal{T}_{\beta^*}^*\{s\} = \frac{1}{\beta^*} \frac{\partial}{\partial s} \left[\ln Z_0\{\beta^*(Js + h/U_{\text{max}})\} \right]. \quad (23)$$

The quantity $f_{\text{MF}}^*\{\beta^*, m\}$ can be understood as the free energy of the mean-field model at inverse temperature β^* with the magnitude of the random field being greater by constant factor $1/U_{\text{max}}$.

As seen in eq. (9), the free energy is the lower limit of the generalized free energy, and therefore eq. (21) leads to

$$f(\beta, m) \geq f_{\text{MF}}\{\beta, m\} + U_{\text{max}} \left\{ \inf_{m(\mathbf{x})} \left\{ \int_{\mathbb{I}^d} d^d x f_{\text{MF}}^*\{\beta^*, m(\mathbf{x})\}, \int_{\mathbb{I}^d} d^d x m(\mathbf{x}) = m \right\} - f_{\text{MF}}^*\{\beta^*, m\} \right\}. \quad (24)$$

By the way, the following equation is known to hold,¹⁵

$$\inf_{m(\mathbf{x})} \left\{ \int_{\mathbb{I}^d} d^d x f_{\text{MF}}^*\{\beta^*, m(\mathbf{x})\}, \int_{\mathbb{I}^d} d^d x m(\mathbf{x}) = m \right\} = C_{\text{env}}\{f_{\text{MF}}^*\{\beta^*, m\}\}, \quad (25)$$

where $C_{\text{env}}\{F; m\}$ is the convex envelope of a function $F(m)$. See Fig. 1.

With eqs. (14), (24) and (25), we obtain

$$f_{\text{MF}}\{\beta, m\} - U_{\text{max}} \Delta f_{\text{MF}}^*\{\beta^*, m\} \leq f\{\beta, m\} \leq f_{\text{MF}}\{\beta, m\}, \quad (26)$$

where the $\Delta f_{\text{MF}}^*\{\beta^*, m\}$ is defined as

$$\Delta f_{\text{MF}}^*\{\beta^*, m\} = f_{\text{MF}}^*\{\beta^*, m\} - C_{\text{env}}\{f_{\text{MF}}^*\{\beta^*, m\}\}. \quad (27)$$

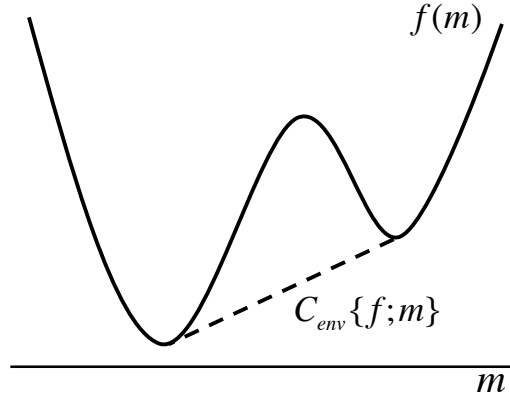


Fig. 1. The function $F(m)$ in bold and its convex envelope $C_{\text{env}}\{F; m\}$ as dashed. Where these two coincide, only the former is drawn.

3.3 Exactness of the mean-field theory (I): Non-conserved magnetization

Let us first analyze the simple case of non-conserved magnetization. Since the Fourier coefficient U_{max} is equal to or smaller than 1, eq. (20) can be simplified as

$$\iint_{\mathbb{I}^d \otimes \mathbb{I}^d} d^d x d^d y U(\mathbf{x} - \mathbf{y}) m(\mathbf{x}) m(\mathbf{y}) \leq \sum_n |m_n|^2 = \int_{\mathbb{I}^d} d^d x m(\mathbf{x})^2. \quad (28)$$

It is not difficult to verify that the above inequality reduces eq. (21) to

$$\mathcal{F}\{\beta; m(\mathbf{x})\} \geq \int_{\mathbb{I}^d} d^d x f_{\text{MF}}\{\beta, m(\mathbf{x})\}. \quad (29)$$

Since the free energy of non-conserved system $f(\beta)$ is the minimum of $\mathcal{F}\{\beta; m(\mathbf{x})\}$ and that of the mean-field theory $f_{\text{MF}}(\beta)$ is the minimum of the above right-hand side, we find

$$f(\beta) \geq f_{\text{MF}}(\beta). \quad (30)$$

It also follows from eq. (14) that $f_{\text{MF}}(\beta)$ is lower-bounded by $f(\beta)$, $f_{\text{MF}}(\beta) \geq f(\beta)$. We therefore conclude the exactness of the mean-field theory,

$$f(\beta) = f_{\text{MF}}(\beta). \quad (31)$$

3.4 Exactness of the mean-field theory (II): Conserved magnetization

Equation (26) indicates that the mean-field theory is exact for systems with conserved magnetization, $f\{\beta, m\} = f_{\text{MF}}\{\beta, m\}$, when the mean-field free energy modified by U_{max} is convex, $\Delta f_{\text{MF}}^*\{\beta^*, m\} = 0$. In particular, a potential in the van der Waals limit has $U(\mathbf{x}) = \delta(\mathbf{x})$

as shown in Appendix C, which means $U_n = 1$ for any n including $U_{\max} = 1$. It follows that $\beta^* = \beta$ and $f_{\text{MF}}^*\{\beta^*, m\} = f_{\text{MF}}\{\beta, m\}$. We therefore find that the mean-field theory is exact when $\Delta f_{\text{MF}}\{\beta, m\} = 0$ for potentials in the van der Waals limit.

For a general potential, the condition of convexity $\Delta f_{\text{MF}}^*\{\beta^*, m\} = 0$ is clearly unsatisfied when the second derivative is negative,

$$\frac{\partial^2}{\partial m^2} f_{\text{MF}}^*\{\beta^*, m\} < 0 \quad (32)$$

as seen in Fig. 1. This fact can also be verified through the relations

$$\left. \frac{\partial^2 \mathcal{F}\{\beta; m(\mathbf{x})\}}{\partial m_{-n'} \partial m_n} \right|_{m(\mathbf{x})=m} = \left(-JU_n + J \frac{\partial \mathcal{T}_\beta^{-1}\{m\}}{\partial m} \right) \delta_{n'n} \quad (33)$$

and

$$\frac{\partial^2}{\partial m^2} f_{\text{MF}}^*\{\beta^*, m\} = \frac{1}{U_{\max}} \left(-JU_{\max} + J \frac{\partial \mathcal{T}_\beta^{-1}\{m\}}{\partial m} \right). \quad (34)$$

See Appendix B for a derivation of eq. (33). These equations indicate that the generalized free energy $\mathcal{F}\{\beta; m(\mathbf{x})\}$ does not take its minimum if the second derivative of $f_{\text{MF}}^*\{\beta^*, m\}$ is negative because eq. (33) becomes negative at $U_n = U_{\max}$.

When the second derivative of $f_{\text{MF}}^*\{\beta^*, m\}$ is positive and $\Delta f_{\text{MF}}^*\{\beta^*, m\} \neq 0$, it is not possible to draw a definite conclusion on the exactness of the mean-field theory.

4. Summary

We have shown the exactness of the mean-field theory for spin systems with long-range interactions. When the magnetization is not conserved, the mean-field theory is exact as long as the interaction potential is in the van der Waals limit or the power of the potential α satisfies $0 \leq \alpha < d$. For systems with conserved magnetization, the mean-field theory is exact for a range of magnetization where the modified mean-field free energy is convex.

These results generalize those of Mori who derived similar conclusions for systems without randomness and for spin-glass cases using the replica method with integer replica number. An advantage of our approach is that we did not use the mathematically ambiguous replica method to treat randomness. It is an important future problem to develop a method to discuss the spin-glass case with long-range interactions without using replicas.

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Appendix A: Variational expression of the free energy

This section derives the variational expression of the free energy in eq. (10). Following closely the method of Mori,^{15–17} we can show that the long-range interaction term in the Hamiltonian is expressed with the coarse-grained variable m_p as

$$\mathcal{H} = \overline{\mathcal{H}} + Ng_1(L, l, \{S_i\}), \quad (\text{A}\cdot 1)$$

where

$$\overline{\mathcal{H}} = -\frac{J}{2} \sum_{p,q=1}^{\Omega} U_{pq} m_p m_q - \sum_i h_i S_i. \quad (\text{A}\cdot 2)$$

The potential U_{pq} in eq. (A.2) is defined as

$$U_{pq} = \sum_{i \in B_p} \sum_{j \in B_q} K(\mathbf{r}_i - \mathbf{r}_j), \quad (\text{A}\cdot 3)$$

and the function $g_1(L, l, \{S_i\})$ converges to zero in the limit $L \rightarrow \infty, l \rightarrow \infty$ as long as we consider the van der Waals limit or the power potential $r^{-\alpha}$ with $0 \leq \alpha < d$.

Let us rewrite the partition function with fixed magnetization m in terms of eq. (A.2),

$$\begin{aligned} Z(\beta, m) &= \text{Tr} \exp(-\beta \mathcal{H}) \delta\left(\frac{\sum_{i=1}^N S_i}{N} - m\right) \\ &= \text{Tr} \int \cdots \int_{\mathbb{R}^{\Omega}} \prod_p dm_p \delta\left(l^d m_p - \sum_{i \in B_p} S_i\right) \delta\left(\frac{\sum_{t=1}^{\Omega} m_t}{\Omega} - m\right) \\ &\quad \times \exp\beta \left(\frac{J}{2} \sum_{q,r} U_{qr} m_q m_r + \sum_i h_i S_i + L^d g_1(L, l, \{S_i\}) \right) \end{aligned} \quad (\text{A}\cdot 4)$$

with the inverse temperature β . The Fourier-transformed expression of the delta function reduces the partition function to

$$Z(\beta, m) = \frac{1}{(2\pi)^{\Omega}} \text{Tr} \int \cdots \int_{\mathbb{R}^{\Omega} \otimes (i\mathbb{R})^{\Omega}} \prod_p dm_p d(\beta J \overline{m}_p) \delta\left(\frac{\sum_{t=1}^{\Omega} m_t}{\Omega} - m\right)$$

$$\times \exp \beta \left(\frac{J}{2} \sum_{q,r} U_{qr} m_q m_r - J \sum_q l^d \bar{m}_q m_q + \sum_q \sum_{i \in B_q} (J \bar{m}_q + h_i) S_i + L^d g_1(L, l, \{S_i\}) \right). \quad (\text{A}\cdot 5)$$

With the definition (12), the trace in eq. (A·5) is rewritten as

$$Z(\beta, m) = \left(\frac{\beta J}{2\pi} \right)^\Omega \int \cdots \int_{\mathbb{R}^\Omega \otimes (i\mathbb{R})^\Omega} \prod_p dm_p d\bar{m}_p \delta \left(\frac{\sum_{t=1}^\Omega m_t}{\Omega} - m \right) \\ \times \exp \left(\frac{\beta J}{2} \sum_{q,r} U_{qr} m_q m_r - \beta J \sum_q l^d \bar{m}_q m_q + \sum_q \sum_{i \in B_q} \ln Z_0 \{ \beta (J \bar{m}_q + h_i) \} + \beta L^d g_1(L, l) \right). \quad (\text{A}\cdot 6)$$

The function $g_1(L, l)$ in eq. (A·6) does not depend on $\{S_i\}$ and tends to zero in the limit $L \rightarrow \infty, l \rightarrow \infty$ for the following reason. The partition function (A·4) can be bounded using the maximum $g_1^{\max}(L, l)$ and the minimum $g_1^{\min}(L, l)$ of $g_1(L, l, \{S_i\})$ among all configuration of spins $\{S_i\}$. Then the trace can be evaluated with eq. (12) if we replace $g_1(L, l, \{S_i\})$ by $g_1^{\min}(L, l)$ or $g_1^{\max}(L, l)$. According to the intermediate value theorem, there is $g_1(L, l)$ such that $g_1^{\min}(L, l) \leq g_1(L, l) \leq g_1^{\max}(L, l)$, using which eq. (A·6) holds. Furthermore, since both of $g_1^{\min}(L, l)$ and $g_1^{\max}(L, l)$ tend to zero in the limit $L \rightarrow \infty, l \rightarrow \infty$, $g_1(L, l)$ also tends to zero.

Now, the law of large numbers is expressed as

$$\sum_{i \in B_p} \ln Z_0 \{ \beta (J \bar{m}_q + h_i) \} = l^d [\ln Z_0 \{ \beta (J \bar{m}_q + h) \}] + l^d g_2(l, \{S_i\}), \quad (\text{A}\cdot 7)$$

where $g_2(l, \{S_i\})$ converges to zero in the limit $l \rightarrow \infty$ as long as the variance of the stochastic variable on the left-hand side is finite. The law of large number originally means

$$\text{Prob} \left\{ \left| \sum_{i \in B_p} \ln Z_0 \{ \beta (J \bar{m}_q + h_i) \} / l^d - [\ln Z_0 \{ \beta (J \bar{m}_q + h) \}] \right| > \epsilon \right\} \rightarrow 0 \quad (\forall \epsilon > 0, l^d \rightarrow \infty), \quad (\text{A}\cdot 8)$$

where $\text{Prob}\{\cdot\}$ is the probability of the condition (\cdot) to occur. Hence, when $l^d \rightarrow \infty$, only the situation where the absolute value in eq. (A·8) is equal to zero can occur. This means that the argument on the left-hand side of eq. (A·8) tends to 0 in the limit $N \rightarrow \infty$. This is equivalent to eq. (A·7). This property can be applied to the partition function (A·6), leading to

$$Z(\beta, m) = \left(\frac{\beta J}{2\pi} \right)^\Omega \int \cdots \int_{\mathbb{R}^\Omega \otimes (i\mathbb{R})^\Omega} \prod_p dm_p d\bar{m}_p \delta \left(\frac{\sum_{t=1}^\Omega m_t}{\Omega} - m \right)$$

$$\exp\left(\frac{\beta J}{2} \sum_{a,r} U_{qr} m_q m_r - \beta J \sum_q l^d \bar{m}_q m_q + \sum_q l^d [\ln Z_0\{\beta(J \bar{m}_q + h)\}] + \beta L^d g(L, l)\right), \quad (\text{A}\cdot 9)$$

where the function $g(L, l)$ is defined as

$$\beta g(L, l) = \beta g_1(L, l) + g_2(l). \quad (\text{A}\cdot 10)$$

For the same reason as before, the dependence of g_2 on S_i can be ignored. With this definition, the function $g(L, l)$ converges to zero in the limit $L \rightarrow \infty, l \rightarrow \infty$ regardless of the state of spins S_i and the distribution of random fields h_i .

Next, we take the continuum limit of space in eq. (A·9),

$$Z(\beta, m) = \left(\frac{\beta J}{2\pi}\right)^\Omega \iint \mathcal{D}m(\mathbf{x}) \mathcal{D}\bar{m}(\mathbf{x}) \delta\left(\int_{\mathbb{I}^d} d^d x m(\mathbf{x}) - m\right) \exp\left(-\beta L^d \mathcal{F}\{\beta; m(\mathbf{x}), \bar{m}(\mathbf{x})\} + \beta L^d g(L, l)\right), \quad (\text{A}\cdot 11)$$

where

$$\begin{aligned} \mathcal{F}\{\beta; m(\mathbf{x}), \bar{m}(\mathbf{x})\} = & -\frac{J}{2} \iint_{\mathbb{I}^d \otimes \mathbb{I}^d} d^d x d^d y U(\mathbf{x} - \mathbf{y}) m(\mathbf{x}) m(\mathbf{y}) \\ & + J \int_{\mathbb{I}^d} d^d x \bar{m}(\mathbf{x}) m(\mathbf{x}) - \frac{1}{\beta} \int_{\mathbb{I}^d} d^d x [\ln Z_0\{\beta(J \bar{m}(\mathbf{x}) + h)\}], \end{aligned} \quad (\text{A}\cdot 12)$$

with $\mathcal{D}m(\mathbf{x}) = \lim_{\Omega \rightarrow \infty} \prod_{p=1}^\Omega dm_p$. When the saddle-point method is applied to $\bar{m}(\mathbf{x})$, the saddle-point equation is given as

$$m(\mathbf{x}) = \mathcal{T}_\beta\{\bar{m}(\mathbf{x})\}. \quad (\text{A}\cdot 13)$$

The function $\mathcal{T}_\beta(x)$ is an increasing function because the definition leads to

$$\frac{\partial}{\partial x} \mathcal{T}_\beta(x) = J\beta \left[\frac{\text{tr} S^2 e^{\beta(Jx+h)S}}{\text{tr} e^{\beta(Jx+h)S}} - \left\{ \frac{\text{tr} S e^{\beta(Jx+h)S}}{\text{tr} e^{\beta(Jx+h)S}} \right\}^2 \right] \geq 0. \quad (\text{A}\cdot 14)$$

Hence the inverse function of $\mathcal{T}_\beta(x)$ can be defined. With the inverse function $\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}$, the generalized free energy $\mathcal{F}\{\beta; m(\mathbf{x})\}$ is expressed as

$$\begin{aligned} \mathcal{F}\{\beta; m(\mathbf{x})\} = & -\frac{J}{2} \iint_{\mathbb{I}^d \otimes \mathbb{I}^d} d^d x d^d y U(\mathbf{x} - \mathbf{y}) m(\mathbf{x}) m(\mathbf{y}) \\ & + J \int_{\mathbb{I}^d} d^d x \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} m(\mathbf{x}) - \frac{1}{\beta} \int_{\mathbb{I}^d} d^d x [\ln Z_0\{\beta(J \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} + h)\}]. \end{aligned} \quad (\text{A}\cdot 15)$$

By applying the saddle point method to $m(\mathbf{x})$, we evaluate the partition function and then

obtain the free energy per spin as

$$f(\beta, m) = \lim_{L, l, L/l \rightarrow \infty} \left(-\frac{1}{\beta L^d} \ln Z(\beta, m) \right) = \inf_{m(\mathbf{x})} \left\{ \mathcal{F}\{\beta; m(\mathbf{x})\}, \int_{\mathbb{I}^d} d^d x m(\mathbf{x}) = m \right\}, \quad (\text{A}\cdot 16)$$

where $\inf\{f, *\}$ means the lower limit of f under the condition $(*)$.

Appendix B: Evaluation of the generalized free energy and its derivative

In this Appendix, we derive eqs. (21) and (33).

First, eq. (21) is evaluated as follows.

$$\begin{aligned} \mathcal{F}\{\beta; m(\mathbf{x})\} &\geq -\frac{J}{2}m^2 - \frac{J}{2}U_{\max} \left\{ \int_{\mathbb{I}^d} d^d x m(\mathbf{x})^2 - m^2 \right\} \\ &+ J \int_{\mathbb{I}^d} d^d x \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}m(\mathbf{x}) - \frac{1}{\beta} \int_{\mathbb{I}^d} d^d x \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} + h)\} \right] \\ &= -\frac{J}{2}m^2 + J\mathcal{T}_\beta^{-1}\{m\}m - \frac{1}{\beta} \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m\} + h)\} \right] \\ &- \frac{J}{2}U_{\max} \int_{\mathbb{I}^d} d^d x m(\mathbf{x})^2 + J \int_{\mathbb{I}^d} d^d x \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}m(\mathbf{x}) - \frac{1}{\beta} \int_{\mathbb{I}^d} d^d x \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} + h)\} \right] \\ &+ \frac{J}{2}U_{\max}m^2 - J\mathcal{T}_\beta^{-1}\{m\}m + \frac{1}{\beta} \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m\} + h)\} \right] \\ &= f_{\text{MF}}\{\beta, m\} \\ &+ U_{\max} \int_{\mathbb{I}^d} d^d x \left\{ -\frac{J}{2}m(\mathbf{x})^2 + J\frac{\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}}{U_{\max}}m(\mathbf{x}) - \frac{1}{\beta U_{\max}} \left[\ln Z_0 \left\{ \beta U_{\max} \left(J\frac{\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}}{U_{\max}} + \frac{h}{U_{\max}} \right) \right\} \right] \right\} \\ &- U_{\max} \left\{ -\frac{J}{2}m^2 + J\frac{\mathcal{T}_\beta^{-1}\{m\}}{U_{\max}}m - \frac{1}{\beta U_{\max}} \left[\ln Z_0 \left\{ \beta U_{\max} \left(J\frac{\mathcal{T}_\beta^{-1}\{m\}}{U_{\max}} + \frac{h}{U_{\max}} \right) \right\} \right] \right\} \\ &= f_{\text{MF}}\{\beta, m\} + U_{\max} \left\{ \int_{\mathbb{I}^d} d^d x f_{\text{MF}}^*\{\beta^*, m(\mathbf{x})\} - f_{\text{MF}}^*\{\beta^*, m\} \right\}. \end{aligned} \quad (\text{B}\cdot 1)$$

Equation (33) is derived as

$$\begin{aligned} \left. \frac{\partial^2 \mathcal{F}\{\beta; m(\mathbf{x})\}}{\partial m_{-n'} \partial m_n} \right|_{m(\mathbf{x})=m} &= \frac{\partial}{\partial m_{-n'}} \left\{ JU_n m_{-n} + J \int_{\mathbb{I}^d} d^d x \frac{\partial \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}}{\partial m(\mathbf{x})} e^{-2\pi i n \cdot x} m(\mathbf{x}) \right. \\ &+ J \int_{\mathbb{I}^d} d^d x \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} e^{-2\pi i n \cdot x} \\ &\left. - \frac{1}{\beta} \int_{\mathbb{I}^d} d^d x \frac{\partial}{\partial \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}} \left[\ln Z_0\{\beta(J\mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} + h)\} \right] \frac{\partial \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}}{\partial m(\mathbf{x})} e^{-2\pi i n \cdot x} \right\} \Big|_{m(\mathbf{x})=m} \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial m_{-n'}} \left\{ JU_n m_{-n} + J \int_{\mathbb{I}^d} d^d x \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\} e^{-2\pi i n \cdot \mathbf{x}} \right\} \Big|_{m(\mathbf{x})=m} \\
&= -JU_n \delta_{n'n} + J \int_{\mathbb{I}^d} d^d x \frac{\partial \mathcal{T}_\beta^{-1}\{m(\mathbf{x})\}}{\partial m(\mathbf{x})} e^{-2\pi i(n-n') \cdot \mathbf{x}} \Big|_{m(\mathbf{x})=m} \\
&= -JU_n \delta_{n'n} + J \frac{\partial \mathcal{T}_\beta^{-1}\{m\}}{\partial m} \int_{\mathbb{I}^d} d^d x e^{-2\pi i(n-n') \cdot \mathbf{x}} \\
&= \left(-JU_n + J \frac{\partial \mathcal{T}_\beta^{-1}\{m\}}{\partial m} \right) \delta_{n'n} \tag{B.2}
\end{aligned}$$

Appendix C: Normalized potential in the van der Waals limit

In this section, we prove that the potential $U(\mathbf{x})$ in the van der Waals limit is the delta function $\delta(\mathbf{x})$ by showing that $(\gamma L)^d \phi(\gamma L \mathbf{x})$ approaches the delta function.

Let the ϵ -vicinity of the origin be written as $B_\epsilon = \{\mathbf{x} \in \mathbb{I}^d : |\mathbf{x}| < \epsilon\}$ and $\mathbb{I}^d \setminus B_\epsilon$ be denoted as \bar{B}_ϵ . The integral of $(\gamma L)^d \phi(\gamma L \mathbf{x})$ over \bar{B}_ϵ is

$$\int_{\bar{B}_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) = \int_{\gamma L \bar{B}_\epsilon} d^d x' \phi(\mathbf{x}'), \tag{C.1}$$

where $\gamma L \bar{B}_\epsilon = \{\mathbf{x} \in (\gamma L)^d \mathbb{I}^d : |\mathbf{x}| \geq \gamma L \epsilon\}$. In the limit $L \rightarrow \infty$, all points in $\gamma L \bar{B}_\epsilon$ tend to ∞ . Hence, for the integrable function $\psi(\mathbf{x})$, the integral (C.1) tends to 0,

$$\lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\gamma L \bar{B}_\epsilon} d^d x' \phi(\mathbf{x}') = 0. \tag{C.2}$$

On the other hand, the integral over \mathbb{I}^d is divided into two parts,

$$1 = \int_{\mathbb{I}^d} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) = \int_{B_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) + \int_{\bar{B}_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}), \tag{C.3}$$

where we have used eq. (8). In the van der Waals limit, the above equation implies

$$\begin{aligned}
\lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\mathbb{I}^d} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) &= \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{B_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) + \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\bar{B}_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) \\
&= \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{B_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) = 1. \tag{C.4}
\end{aligned}$$

Next, let us prepare a test function $f(\mathbf{x})$ that is continuous and integrable, and evaluate the integral of $f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x})$. If we can prove the following equation,

$$\lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\mathbb{I}^d} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) = f(\mathbf{0}), \tag{C.5}$$

$(\gamma L)^d \phi(\gamma L \mathbf{x})$ is confirmed to approach the delta function. The integral of $f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x})$ is

divided in two parts as in eq. (C-3),

$$\int_{\mathbb{I}^d} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) = \int_{B_\epsilon} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) + \int_{\overline{B_\epsilon}} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}). \quad (\text{C-6})$$

The first term on the right-hand side of eq. (C-6) is bounded as

$$\left(\inf_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}') \right) \cdot \int_{B_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) \leq \int_{B_\epsilon} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) \leq \left(\sup_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}') \right) \cdot \int_{B_\epsilon} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}). \quad (\text{C-7})$$

In the van der Waals limit, the above equation is reduced to, according to eq. (C-4).

$$\inf_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}') \leq \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{B_\epsilon} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) \leq \sup_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}'). \quad (\text{C-8})$$

The second term on the right-hand side of eq. (C-6) is evaluated as

$$\begin{aligned} \inf_{\mathbf{x}' \in \overline{B_\epsilon}} \{f(\mathbf{x}') - f(\mathbf{0})\} \int_{\overline{B_\epsilon}} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}) &\leq \int_{\overline{B_\epsilon}} d^d x \{f(\mathbf{x}) - f(\mathbf{0})\} (\gamma L)^d \phi(\gamma L \mathbf{x}) \\ &\leq \sup_{\mathbf{x}' \in \overline{B_\epsilon}} \{f(\mathbf{x}') - f(\mathbf{0})\} \int_{\overline{B_\epsilon}} d^d x (\gamma L)^d \phi(\gamma L \mathbf{x}). \end{aligned} \quad (\text{C-9})$$

In the van der Waals limit, the right- and left-hand sides both tend to 0 due to eq. (C-2), which leads to

$$\lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\overline{B_\epsilon}} d^d x \{f(\mathbf{x}) - f(\mathbf{0})\} (\gamma L)^d \phi(\gamma L \mathbf{x}) = 0. \quad (\text{C-10})$$

With eqs. (C-6), (C-8), and (C-10), we find

$$\inf_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}') \leq \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\mathbb{I}^d} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) \leq \sup_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}'). \quad (\text{C-11})$$

The above inequality is reduced to

$$\begin{aligned} \left| \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\mathbb{I}^d} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) - f(\mathbf{0}) \right| &\leq \max \left\{ \left| \inf_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}') - f(\mathbf{0}) \right|, \left| \sup_{\mathbf{x}' \in B_\epsilon} f(\mathbf{x}') - f(\mathbf{0}) \right| \right\} \\ &\leq \sup_{\mathbf{x}' \in B_\epsilon} |f(\mathbf{x}') - f(\mathbf{0})|. \end{aligned} \quad (\text{C-12})$$

Next, let us remember that the function $f(\mathbf{x})$ is continuous:

$$\sup_{\mathbf{x} \in B_\delta} |f(\mathbf{x}) - f(\mathbf{0})| < \epsilon' \quad (\forall \epsilon', \exists \delta > 0). \quad (\text{C-13})$$

With eqs. (C-12) and (C-13), we can state there is δ such that

$$\left| \lim_{\gamma \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\mathbb{I}^d} d^d x f(\mathbf{x})(\gamma L)^d \phi(\gamma L \mathbf{x}) - f(\mathbf{0}) \right| < \epsilon \quad (\forall \epsilon > 0), \quad (\text{C-14})$$

which means eq. (C-5).

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