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Exact Results on the Ising Spin Glass in Finite Dimensions

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The distribution functions of internal fields and local internal energy are calculated exactly on a line in the phase diagram of the Ising spin glass on an arbitrary lattice. The results exhibit no singular behavior. Independence of distributions at different locations is proved.

Recent Monte Carlo simulations[1),2)] have revived high interest in the possibilities of a spin glass phase as a thermodynamic state in three dimensions. Exact results should assist us to further understand the properties of finite-dimensional random Ising models. The exact internal energy and other useful results are available[3)~5)] on a line in the phase diagram. In this paper we calculate, without approximations, the distributions functions of internal fields and local internal energy on the same line. The lattice structure or spatial dimensionality is irrelevant to the following argument.

Let us begin with the case of the Gaussian distribution of randomness. The distribution function of an internal field at site 0 is defined by

$$\begin{aligned} P(h)=&\int_{-\infty}^{\infty}[\prod dJ_{ij}G(J_{ij})]\sum_{\{S=\pm1\}}\\ &\times\delta(h-\sum_m J_{0m}S_m)\\ &\times\exp(\beta\sum J_{ij}S_iS_j)/Z(\{J_{ij}\})\\ \equiv&\langle\delta(h-\sum_m J_{0m}S_m)\rangle\,, \end{aligned}\tag{1}$$

where

$$G(J_{ij})=\frac{1}{\sqrt{2\pi}J}\exp\{-(J_{ij}-J_0)^2/2J^2\}$$

and

$$Z(\{J_{ij}\})=\sum_{\{S=\pm1\}}\exp(\beta\sum J_{ij}S_iS_j)$$

β is $1/k_BT$. The global up-down symmetry of the Ising system yields

$$\begin{aligned} P(h)=&\langle\delta(h-\sum J_{0m}S_m)\rangle=\langle\delta(h+\sum J_{0m}S_m)\rangle\\ =&\frac{1}{2}\langle\delta(h-\sum J_{0m}S_m)\\ &+\delta(h+\sum J_{0m}S_m)\rangle\,. \end{aligned}\tag{2}$$

Since the last expression is manifestly gauge-invariant, the method of Ref. 3) can be applied to the calculation of the average. Technical details are found in Ref. 3) and we only outline the method. A gauge transformation $J_{ij}\to\sigma_i\sigma_jJ_{ij}$ ($\sigma=\pm1$) is performed, a sum over $\{\sigma\}$ is taken, and the result is divided by 2^N (N is the number of sites). When $\beta J^2=J_0$ (which defines a hyperbolic curve in the phase diagram[3)]), $P(h)$ reduces to a simple form

$$\begin{aligned} P(h)=&2^{-N}\int_{-\infty}^{\infty}[\prod dJ_{ij}G_0(J_{ij})]\\ &\times\sum_{\{S\}}\exp(\beta\sum J_{ij}S_iS_j)\\ &\times\frac{1}{2}\{\delta(h-\sum J_{0m}S_m)\\ &+\delta(h+\sum J_{0m}S_m)\}\,, \end{aligned}\tag{3}$$

where

$$G_0(J_{ij})=\frac{1}{\sqrt{2\pi}J}\exp\{-(J_{ij}^2+J_0^2)/2J^2\}\,.$$

It is easy to carry out the integration over $\{J_{ij}\}$ in (3) except those related to site 0. In the resulting relation we perform a gauge transformation $J_{0m}\to J_{0m}S_m$ and the summation over $\{S\}$ to yield

$$\begin{aligned} P(h)=&(\sqrt{2\pi}J)^{-z}\exp(-\beta^2J^2z/2)\cosh\beta h\\ &\times\int_{-\infty}^{\infty}\prod dJ_{0m}\exp(-\sum J_{0m}^2/2J^2)\\ &\times\delta(h-\sum J_{0m})\,, \end{aligned}$$

where z is the number of nearest neighbors. After integration we find

$$\begin{aligned} P(h)=&\frac{1}{2\sqrt{2\pi z}J}[\exp\{-(h-z\beta J^2)^2/2zJ^2\}\\ &+\exp\{-(h+z\beta J^2)^2/2zJ^2\}]\,. \end{aligned}$$

This is the final result. The above argument is easily applied to the case of the binary randomness

$$p\delta(J_{ij}-J)+(1-p)\delta(J_{ij}+J)\,.$$

On the line $\tanh\beta J=2p-1$ in the phase diagram[3)] we have

$$P(h) \propto \cosh \beta h \sum_{\{\tau=\pm 1\}} \delta(h - \sum \tau_{0m}) .$$

The proportionality constant is the normalization factor independent of h.

Another quantity of interest is the two-site correlation

$$P_2(h_1, h_2) = \langle \delta(h_1 - |\sum_m J_{1m} S_m|) \times \delta(h_2 - |\sum_l J_{2l} S_l|) \rangle . \quad (4)$$

If the set $\{J_{1m}\}$ has no common element with $\{J_{2l}\}$ (i.e., site 1 is not a nearest neighbor of 2), the techniques for the single-site distribution readily apply to (4). The calculations are performed at sites 1 and 2 independently and the result is

$$P_2(h_1, h_2) = 4 P_+(h_1) P_+(h_2) ,$$

where $P_+(h)$ is defined by restricting $P(h)$ to the region $h>0$. Thus the distributions of the absolute values at two distinct sites are independent unless the concerned sites are nearest neighbors. The signs of internal fields at two sites should be correlated because the low-temperature tail of the present line enters the ferromagnetic phase.[4]

The distribution function of local internal energy is evaluated by the same method. For the Gaussian case, the answer is

$$P(E) = \langle \delta(E - J_{ij} S_i S_j) \rangle = \frac{1}{\sqrt{2\pi} J} \exp\{-(E - \beta J^2)^2 / 2J^2\} , \quad (5)$$

where $\beta J^2 = J_0$. The binary distribution yields

$$P(E) = \{ e^{\beta J} \delta(E - J) + e^{-\beta J} \delta(E + J) \} / 2 \cosh \beta J , \quad (6)$$

if $\tanh \beta J = 2p - 1$. On the same line ($\beta J^2 = J_0$ or $\tanh \beta J = 2p - 1$) the two-bond correlation

$$P_2(E_1, E_2) = \langle \delta(E_1 - J_{ij} S_i S_j) \times \delta(E_2 - J_{lm} S_l S_m) \rangle$$

is decomposed to a product $P(E_1) P(E_2)$ of the single-bond distributions. This independence of energy distributions is a consequence of the independence of $\{J_{ij}\}$ at different bonds. The nontrivial fact is that the latter independence is reflected to the former one only on a special line in the phase diagram.

In this paper we have pointed out that the method of Ref. 3) can be applied to the calculation of internal fields and local energy which are much more microscopic quantities than the total energy and other global quantities obtained before.[3] We have also proved independence of local internal energy at different bonds, which indicates that the system effectively splits into uncorrelated set of bonds on the present line in the phase diagram. This is the reason why the total energy shows no trace of cooperative effects[3] even though the line intersects a phase boundary.[4] Here one should not forget that this independence holds only for the local quantities. The specific heat and the susceptibility on the same line would reflect global spin fluctuations to result in singular behavior at the critical point (although there is no way at present to explicitly demonstrate singularities in these functions). This difference between local and global variables is an important feature of the present line.

As discussed in Ref. 3), if there exists a spin glass phase (as expected in three dimensions[1),2)]), the tricritical point (where the paramagnetic, ferromagnetic and spin glass phases merge) is likely to be located on the present line. Thus it is of interest to perform Monte Carlo simulations along the line to accurately find out the position of the tricritical point in the phase diagram. The results in this paper are useful to confirm that the simulated system is in thermal equilibrium: If the observed distribution of internal field obeys the present exact result, one can safely assume that equilibrium has been reached at a microscopic level. Numerical calculations are in progress.

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