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Geometry-Induced Phase Transition in the $\pm J$ Ising Model

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We present an argument for the existence of a geometry-induced phase transition in the $\pm J$ Ising model on an arbitrary lattice ($d \geq 2$) with asymmetric probability of ferromagnetic and antiferromagnetic bonds. This transition manifests itself as a vertical phase boundary separating, possibly, a ferromagnetic and a non-ferromagnetic (paramagnetic or spin glass) phases.

Systematic theoretical analysis of spin glasses was initiated by Sherrington and Kirkpatrick[1] who proposed and solved an infinite-range Ising model with Gaussian randomness in exchange interactions. Incompleteness of their solution was later remedied by a replica symmetry breaking scheme,[2] leading to a modified phase diagram of the infinite-range model.[3] Efforts to investigate short-range models in finite dimensions have been mainly done in numerical studies such as the transfer matrix method[4] and Monte Carlo simulations.[5] From these numerical calculations it is now believed that the two-dimensional $\pm J$ Ising model does not have a spin glass transition whereas the three-dimensional model does have one. These results on short-range models are, however, limited to the case of equal probability of $+J$ and $-J$ bonds, and we are still far away from drawing the whole phase diagram for unequal distribution in finite dimensions. The present Letter represents a step towards this goal. More precisely, we apply the method of gauge transformation[6] to provide an argument for the existence of a vertical phase boundary in the phase diagram of the finite-dimensional $\pm J$ Ising model. This vertical phase boundary is shown to be a manifestation of a singularity in the entropy of distribution of frustration.

The configurational average of free energy is conveniently written as a weighted sum as[6]

$$-\beta F = (2\cosh K_p)^{-N_B} \sum_{\{\tau_{ij}=\pm 1\}} \exp\Big(K_p \sum_{\langle ij\rangle} \tau_{ij}\Big) \times \ln \sum_{\{S_i=\pm 1\}} \exp\Big(K \sum_{\langle ij\rangle} \tau_{ij} S_i S_j\Big), \tag{1}$$

where N_B is the number of bonds, $K=\beta J$ and K_p is defined by

$$\exp(-2K_p) = \frac{1-p}{p}, \tag{2}$$

with p the probability of $J_{ij}=J(>0)$. We apply a gauge transformation

$$\tau_{ij} \to \tau_{ij}\sigma_i\sigma_j, \quad S_i \to S_i\sigma_i, \tag{3}$$

sum over all possible $\{\sigma_i\}$ and divide by 2^N to find

$$-\beta F = 2^{-N}(2\cosh K_p)^{-N_B} \times \sum_{\{\tau\}} \sum_{\{\sigma_i=\pm 1\}} \exp(K_p \Sigma \tau_{ij}\sigma_i\sigma_j) \times \ln \sum_{\{S\}} \exp(K\Sigma\tau_{ij}S_iS_j). \tag{4}$$

If we write

$$Z(\{\tau_{ij}\}, K) = \sum_{\{S\}} \exp(K\Sigma\tau_{ij}S_iS_j), \tag{5}$$

(4) is expressed as

$$-\beta F = 2^{-N}(2\cosh K_p)^{-N_B} \sum_{\{\tau\}} Z(\{\tau_{ij}\}, K_p) \times \ln Z(\{\tau_{ij}\}, K). \tag{6}$$

Since the gauge transformation (3) leaves the distribution of frustration invariant, we learn by comparison of (1) with (6) that the sum of probabilities of various bond configurations with the same distribution of frustration is written as the partition function of the same $\pm J$ Ising model with effective coupling K_p.

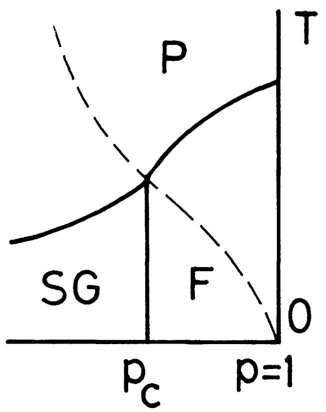

Fig. 1. The phase diagram of the $\pm J$ Ising model. The dashed line corresponds to the condition $K=K_p$. The vertical phase boundary at $p=p_c$ represents the geometry-induced phase transition. The ferromagnetic phase is denoted by F, paramagnetic phase by P and spin glass phase by SG. The spin glass phase may not be present in low dimensions.

Thus the partition function (5) with K replaced by K_p is interpreted as the probability weight of a given distribution of frustration (with a trivial normalization factor neglected). Let us now consider the case $K=K_p$ which defines a line in the phase diagram (Fig. 1).[6)] It is apparent from topological considerations that the line intersects a phase boundary between the paramagnetic and ferromagnetic phases. (It is known[6)] that this line does not enter the spin glass phase if any.) At the point of intersection the free energy has a singularity as a function of $K(=K_p)$. (The internal energy does not have a singularity at the same point,[6)] but this fact never implies the absence of a singularity in other thermodynamic quantities.) Then (6) with $K=K_p$ is singular at some $K_p=K_{pc}$ (or $p=p_c$). On the other hand (6) with $K=K_p$ may also be seen as the entropy of frustration (neglecting some trivial terms corresponding to the normalization of probability) because it represents the average of logarithm of probability of frustration. Thus the entropy of frustration is singular at $K_p=K_{pc}$. But the entropy of frustration is originally of geometrical character (i.e. not defined in terms of thermal variables, spins, of the system). Thus the existence of a singularity in the entropy of frustration is not limited to a specific temperature. In other words a "geometrical anomaly" of the distribution of frustration exists at $K_p=K_{pc}$ independent of the temperature. This geometrical anomaly should strongly affect the behavior of thermodynamic quantities of the system because the distribution of frustration determines a background on which thermal variables (spins) develop time evolution. The singular behavior of the free energy on the line $K=K_p$ at $K_p=K_{pc}$ is regarded as a manifest example of this geometry-induced singularity. Of course, if the temperature is higher than some critical value T_0, no singularity will show up because of thermal disturbance. The geometry-induced phase transition at $K_p=K_{pc}$ ($p=p_c$) for $T\leq T_0$ corresponds to a vertical phase boundary as depicted in Fig. 1.

In two dimensions various numerical studies[7)] suggest an almost vertical boundary between a ferromagnetic and a non-ferromagnetic phases. The present argument establishes that the boundary is strictly vertical. In the infinite-range case the diagram drawn by Toulouse[3)] has a vertical boundary between a spin glass and a ferromagnetic phases, which is consistent with the present result. (Note that the $\pm J$ model reduces to a Gaussian-distributed model treated in ref. 3 in the infinite-range limit.[8)]) It is useful to remember that the line $K=K_p$ intersects the paramagnetic-ferromagnetic boundary at a point where the ferromagnetic phase has the smallest p (i.e. the "left-most" point in the ferromagnetic phase).[6)] Therefore we conclude that a vertical boundary extends downwards from the left-most point of the ferromagnetic phase in any dimension (two or larger), leading to a phase diagram in Fig. 1.

If the spin variable is not of Ising type, the effect of geometrical anomaly on thermal variables would be more indirect than in the Ising case. Nevertheless the m-vector model in the infinite-range limit has a vertical boundary at the same location as in the infinite-range Ising (SK) model.[9)] This fact may suggest a universal vertical boundary independent of the spin symmetry (depending only on the lattice structure). If the spin variable does not have the basic $\pm$ symmetry, as in the Potts model,[10)] the relation between frustration and thermodynamic quantities should be quite different and the above argument does not apply.

References

1) D. Sherrington and S. Kirkpatrick: Phys. Rev. Lett. **35** (1975) 1792.
2) G. Parisi: Phys. Rev. Lett. **43** (1979) 1545; J. Phys. A**13** (1980) L115, 1101, 1887.
3) G. Toulouse: J. Phys. (France) **41** (1980) L447.
4) I. Morgenstern and K. Binder: Phys. Rev. Lett. **43** (1979) 1615; Phys. Rev. B**22** (1980) 288; Z. Phys. B**39** (1980) 227.
5) R. N. Bhatt and A. P. Young: Phys. Rev. Lett. **54** (1985) 924; A. T. Ogielski and I. Morgenstern: Phys. Rev. Lett. **54** (1985) 928.
6) H. Nishimori: Prog. Theor. Phys. **66** (1981) 1169.
7) See H. Morita: J. Phys. C**16** (1983) 181.
8) J. L. van Hemmen and A. Sütő: J. Phys. (France) **45** (1984) 1277.
9) M. Gabay and G. Toulouse: Phys. Rev. Lett. **47** (1981) 201.
10) D. Elderfield and D. Sherrington: J. Phys. C**16** (1983) L971.