Journal of the Physical Society of Japan
Vol. 56, No. 4, April, 1987, pp. 1568–1576

# Phase Diagram and Critical Exponents of the $\pm J$ Ising Model in Finite Dimensions by Monte Carlo Renormalization Group

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(Received December 15, 1986)

We have performed large-scale numerical simulations for the $\pm J$ Ising model on the simple cubic and the square lattices with asymmetric weight of ferromagnetic and antiferromagnetic bonds. The Monte Carlo renormalization group method is used to estimate $T_c$, $\gamma/\nu$, $\beta/\nu$ and $\nu$ at various points along the phase boundary between ferromagnetic and paramagnetic phases including the tricritical point. The obtained values of $T_c$ and the resulting phase diagrams of the simple cubic and the square lattices are much more accurate than those estimated so far. In the simple cubic lattice, we find evidence for the possibility that universality of the ferromagnetic critical exponents holds along the critical line except at the tricritical point, while weak universality holds even at the tricritical point. Weak universality seems to hold also in the square lattice.

## §1. Introduction

In the last decade many efforts have been directed to understanding the spin glass (SG) state of random magnets. At the level of mean field theory, which was first applied to the SG problem by Edwards and Anderson[1] and by Sherrington and Kirkpatrick,[2] a unified picture seems to have been established through the replica-symmetry-breaking theory of Parisi.[3,4] On the other hand, extensive investigation of finite dimensional systems has become active only recently. For instance, the existence of a finite critical temperature of the symmetric $\pm J$ Ising model on the $3d$ simple cubic lattice was established as $T_c \simeq 1.2$ by careful Monte Carlo (MC) simulations.[5,6] Singh and Chakravarty[7] studied this model by long series of high temperature expansions in 2, 3 and 4-dimensional spaces. Their analysis strongly indicates that the lower critical dimension $d_{lc}$ of this model falls in the range $2 \leq d_{lc} \leq 3$. Those results on the finite dimensional systems are only for the $\pm J$ Ising model with symmetric weight of ferromagnetic and antiferromagnetic bonds, and systematic efforts to draw a precise phase diagram of the asymmetric case have been scarce, particularly in three dimensions. To name one of the recent studies, Reger and Zippelius[8] carried out a dynamical high temperature expansion in the whole region of the phase diagram. However, since their main interest was in dynamical properties, problems such as the precise determination of the phase diagram and ferromagnetic critical exponents still remain unsolved. In this situation we have performed large-scale Monte Carlo renormalization group (MCRG) studies of the asymmetric $\pm J$ model on the $3d$ simple cubic and the $2d$ square lattices.

Our particular interest has been in the tricritical region where the paramagnetic, ferromagnetic and SG (if any) phases merge. In this region there are a few analytical results which serve as useful guides in performing simulations. As has been shown by one of the authors (H. N.),[9–11] it is convenient to consider a special line in the phase diagram defined by $\exp(-2\beta J) = (1-p)/p$ where $p$ is the probability weight of ferromagnetic bonds. We refer to this line as the crossover line hereafter.* According to the geometrical arguments,[9] this line must cross the phase

* We have given this name because two types of critical behavior (induced by thermal and configurational scaling fields, respectively) cross over on this line as ex-

boundary of paramagnetic and ferromagnetic phases, and there are indications that this point of intersection is the tricritical point. The total energy can be calculated exactly on the crossover line on any lattice and the result is a simple analytic function of $p$. Also a few useful relations can be proved for the spin correlations. However, magnetization and susceptibility have not been obtained analytically, both of which should exhibit singular behavior at the (tri-) critical point. Therefore, as analytical results do not yield direct information on the detailed shapes of phase boundaries, we have performed MC simulations along the crossover line as well as, in the case of $3d$, at a few fixed values of $p$ ($=0.83$ and 0.90). The Monte Carlo renormalization group method[12] has been used to determine the critical points and ferromagnetic critical exponents.*

In the next section we explain the method of our simulation and its analysis. In §3 numerical results are presented as well as discussions. A few remarks are made in the last section.

## §2. Methods of Simulation and Its Analysis

We treat the $\pm J$ Ising model with asymmetric probability weight on the simple cubic and the square lattices with fully periodic boundary conditions. Our model Hamiltonian is

$$H=-\sum J_{ij}S_iS_j \quad (S_i=\pm 1), \tag{2.1}$$

where the interaction $J_{ij}$ is only in the nearest-neighbor bonds and is randomly distributed independently at each bond with the probability

$$P(J_{ij})=p\delta(J_{ij}-J)+(1-p)\delta(J_{ij}+J). \tag{2.2}$$

We measure the temperature in units of $J/k_B$ hereafter.

Each simulation is started from a random spin configuration. The conditions of the simulation on the simple cubic lattice are as follows. The size of the system is $32\times 32\times 32$. On the crossover line, the simulation has been made up to $1.6\times 10^6$ Monte Carlo steps (MCS) per spin, and data for the last $4\times 10^5$ MCS/spin have been used to calculate the thermal averages. For fixed $p$($=0.83$ and 0.90), the total number of steps was $4\times 10^5$ MCS/spin and the initial $2\times 10^5$ MCS/spin were discarded for equilibration. Ten values of the parameters ($T$ or $p$) have been selected for each line to be simulated. That is, we took ten points in the range of the temperature $1.379\leq T\leq 2.118$ ($0.72\leq p\leq 0.81$) on the crossover line, and similarly ten points in $2.3\leq T\leq 3.2$ for $p=0.83$, and $3.0\leq T\leq 3.9$ for $p=0.90$. As for the square lattice, the size of the system is $64\times 64$, the simulation has been made up to $3.5\times 10^6$ MCS/spin, and the range of the temperature is $0.8192\leq T\leq 1.216$ ($0.83\leq p\leq 0.92$) on the crossover line. All calculations were carried out on a vector processor HITAC S810/20 at the Computer Center of the University of Tokyo.

We use the MCRG method proposed by Kikuchi and Okabe[12] to determine the critical points and critical exponents. First, we explain the method in the case of the non-random system. The relevant scaling fields are the temperature-like one $u_T$, and the field-like one $u_H$. We divide a spin system into blocks of scale factor $b$ (in the present calculation, it is chosen as $b=2$), and define block spins by a simple majority rule. (Note that this simple block transformation to look for a ferromagnetic transition can be applied to the present random system because our lines, the crossover line and the $p$-fixed lines, never enter the SG phase.[9]) In the critical region, the free energy density in a $d$-dimensional space is transformed by the block spin transformation as

$$f(u_T, u_H)=b^{-d}f(b^{y_T}u_T, b^{y_H}u_H), \tag{2.3}$$

where $y_T$ and $y_H$ are the exponents corresponding to $u_T$ and $u_H$, respectively. Making use of eq. (2.3), the magnetization and susceptibility are found to transform as

$$M(u_T, u_H)=b^{y_H-d}M(b^{y_T}u_T, b^{y_H}u_H), \tag{2.4}$$

$$\chi(u_T, u_H)=b^{2y_H-d}\chi(b^{y_T}u_T, b^{y_H}u_H). \tag{2.5}$$

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plained later. Another reason for this name is in the relation $m=Q$ (ferromagnetic long-range order equals spin glass order) established on the line.[9] It is thus highly probable that $m\geq Q$ above the line (see Fig. 8) while $m\leq Q$ below, which implies that the ferromagnetic and spin glass orders exchange their dominance on this line.

* We note here that our calculations avoid the SG region and treats only the transition between ferromagnetic and paramagnetic phases.

We hereafter set $H=0+$ and omit the argument $u_H$ from $f$, $M$ and $\chi$. This block transformation is repeated $n$ times, and let $\chi_n$ and $M_n$ denote the susceptibility and magnetization, respectively, of the block spin system after $n$ transformations. These quantities are related to those of the original system as

$$\chi_n(u_T)=\chi(b^{ny_T}u_T), \tag{2.6}$$

$$M_n(u_T)=M(b^{ny_T}u_T). \tag{2.7}$$

It is convenient to introduce the following function to determine the critical temperature $T_c$ and critical exponents:

$$\Delta_{nm}(T)\equiv\frac{\log[\chi_n(u_T)/\chi_m(u_T)]}{\log[b^n/b^m]}. \tag{2.8}$$

From eqs. (2.5) and (2.6), we see that $\Delta_{nm}(T)$ is a constant independent of the set $[n, m]$ if the original system is quite near the criticality (which justifies the scaling ansatz (2.5)):

$$\Delta_{nm}(T)=d-2y_H=-\frac{\gamma}{\nu}. \tag{2.9}$$

The same argument holds for $M_n$:

$$\Gamma_{nm}(T)\equiv\frac{\log[M_n/M_m]}{\log[b^n/b^m]}=d-y_H=\frac{\beta}{\nu}. \tag{2.10}$$

Since we have numerical values of $\chi_n$, $\chi_m$, $M_n$ and $M_m$, it is possible to estimate $T_c$, $\gamma/\nu$ and $\beta/\nu$ from MCRG.

Also the exponent $\nu$ can be obtained as follows. From eq. (2.6), the singular part of the renormalized susceptibility $\chi_n(u_T)$ takes the form

$$\chi_n(u_T)=\chi(b^{ny_T}u_T)=u_T^{-\gamma}b^{-ny_T\gamma}\bar{\chi}(b^{ny_T}u_T), \tag{2.11}$$

where $\bar{\chi}(x)$ is an analytic function of $x$ and can be expanded in powers of $x$ as

$$\bar{\chi}\simeq\bar{\chi}_0+\bar{\chi}_1x+\cdots. \tag{2.12}$$

From eqs. (2.8), (2.9) and (2.12), we obtain

$$\Delta_{nm}(T)\simeq-\frac{\gamma}{\nu}+\left(\frac{\bar{\chi}_1}{\bar{\chi}_0}\right)\frac{(b^{n/\nu}-b^{m/\nu})}{(n-m)\log b}u_T, \tag{2.13}$$

for $|b^{n/\nu}u_T|$, $|b^{m/\nu}u_T|\ll1$. (Equation (2.9) is the limiting case of $u_T\to0$ in eq. (2.13).) In order to estimate the exponent $\nu$ from the MCRG data, we have to suppress the unknown constant $(\bar{\chi}_1/\bar{\chi}_0)$ in eq. (2.13). To do this, we numerically calculate the ratio

$$\frac{\left[\frac{\partial\Delta_{nm}}{\partial T}\right]_{T=T_c}}{\left[\frac{\partial\Delta_{n'm'}}{\partial T}\right]_{T=T_c}}=\frac{(b^{n/\nu}-b^{m/\nu})/(n-m)}{(b^{n'/\nu}-b^{m'/\nu})/(n'-m')}, \tag{2.14}$$

from which $\nu$ is estimated since we know the values of $b$, $[n, m]$ and $[n', m']$.

The above arguments are developed for a non-random system, and some modifications are necessary in applying them to random cases. In our random system, one more relevant scaling field $u_p$ (corresponding to $p$) must be considered. Thus we write $\chi_n(u_T, u_p)$, $\chi(b^{y_T}u_T, b^{y_p}u_p)$ etc. instead of $\chi_n(u_T)$, $\chi(b^{y_T}u_T)$ etc. in eqs. (2.4)-(2.7). Even in the random case with extra scaling field $u_p$, the first equality of eq. (2.9) and the second one of eq. (2.10) are clearly valid precisely at a critical point because all scaling fields vanish there (and thus the extra field $u_p$ plays no part). Therefore $T_c$, $p_c$, $2y_H-d$ and $y_H-d$ can be estimated from the MCRG data of the random system simply by using eqs. (2.9) to (2.10). Next we have to relate $y_H$ with $\gamma$ and $\beta$. The scaling form in the random system corresponding to eq. (2.5) is

$$\chi(u_T, u_p)=b^{2y_H-d}\chi(b^{y_T}u_T, b^{y_p}u_p). \tag{2.15}$$

Setting $b=u_T^{-1/y_T}$, we obtain

$$\chi(u_T, u_p)=u_T^{-(2y_H-d)/y_T}\chi(1, s), \tag{2.16}$$

where $s\equiv u_p/u_T^{\phi}$ and $\phi\equiv y_p/y_T$ (the crossover exponent). If $s$ is finite (or zero) in the limit $u_T\to0$ and $u_p\to0$, $\chi(1, s)$ is an analytic function (approaching a constant) and $\chi(u_T, u_p)$ diverges like $u_T^{-(2y_H-d)/y_T}$. Then, by definition of $\gamma$, we obtain the relation $\gamma=(2y_H-d)/y_T$. If $s$ goes to infinity, $\chi(1, s)$ would behave in a singular manner and the leading term of $\chi(u_T, u_p)$ may not be $u_T^{-(2y_H-d)/y_T}$. The above two cases cross over at $s\sim O(1)$, namely,

$$u_p/u_T^{\phi}=\text{const.} \tag{2.17}$$

(It is remarked that the line defined by eq. (2.17) corresponds to an invariant line under RG transformation near the critical point in the following sense. Assume that a point $(u_T, u_p)$ in the critical region is transformed to another point $(b^{y_T}u_T, b^{y_p}u_p)$ by an RG transfor-

mation. If both the original and the transformed points are on the same line in the phase diagram, i.e. $u_p=g(u_T)$ and $b^{y_p}u_p=g(b^{y_T}u_T)$, we call it an invariant line. It is easy to check that eq. (2.17) defines an invariant line.) Similar arguments hold for any thermodynamic quantity with singular behavior near the critical point. For example, one can derive $\nu=1/y_T$ from the scaling of the correlation length and $\beta=(d-y_H)/y_T$ from the magnetization, both in the region $s\lesssim O(1)$. Consequently, the exponents $2y_H-d$ and $d-y_H$ obtained by the MCRG method for the random system are identified as $\gamma/\nu$ and $\beta/\nu$, respectively, in the region $u_p/u_T^{\phi}\lesssim O(1)$ (see Fig. 1).

On the other hand, eqs. (2.11)-(2.14) for the estimation of $\nu$ cannot always be used in the random system, because $\bar{\chi}_1/\bar{\chi}_0$ in eq. (2.13) generally depends on $u_p$. Nevertheless, the following two cases are the exceptions. In the first case the data points are on the invariant line of an RG transformation such as $u_p/u_T^{\phi}=c$. In this case, the susceptibility $\chi(u_T, u_p)=\chi(u_T, cu_T^{\phi})$ satisfies

$$\chi(u_T, cu_T^{\phi})=b^{2y_H-d}\chi(b^{y_T}u_T, c\{b^{y_T}u_T\}^{\phi}), \tag{2.18}$$

in the critical region. Since $\chi$ here can be regarded as a function of a single variable $u_T$, this is the same scaling relation as that of non-random case. Therefore, we can derive the same relations as eqs. (2.11)-(2.14) for $\chi(u_T, cu_T^{\varphi})$. The second case is that the data points are on the line $p=$const, on which $u_p/u_T^{\phi}\sim 0$ near the critical point. In analogy with eqs. (2.11) and (2.16), $\chi(u_T, u_p)$ takes the form

$$\chi(u_T, u_p)=u_T^{-\gamma}\bar{\chi}(u_T, u_p/u_T^{\phi}), \tag{2.19}$$

where $\bar{\chi}(u_T, u_p/u_T^{\phi})$ is an analytic function of $u_T$ and $u_p/u_T^{\phi}$. The first argument of this function denotes the closeness to the critical point and the second one indicates the direction from the criticality (see Fig. 1). In the case of $p=$const, since $u_p/u_T^{\phi}$ is negligible as compared with $u_T$, $\bar{\chi}(u_T, u_p/u_T^{\phi})$ can be expanded as

$$\bar{\chi}\simeq\bar{\chi}_{00}+\bar{\chi}_{10}u_T+\cdots, \tag{2.20}$$

which is similar to eq. (2.12) and the argument following it is valid.

It is useful to note here that the crossover line is an invariant line under an RG transformation as follows. According to ref. 9, the free energy of the random system is expressed as

$$-\beta F(K, K_p)=\frac{2^{-N}}{(2\cosh K_p)^{N_b}}\sum_{\{\tau_{ij}\}}Z(K_p, \{\tau\})\times\ln Z(K, \{\tau\}), \tag{2.21}$$

$$Z(K, \{\tau\})\equiv\sum_{\{S\}}\exp(K\sum_{(ij)}\tau_{ij}S_iS_j), \tag{2.22}$$

where $K=J/k_BT$ and $\exp(-2K_p)=(1-p)/p$. $N_b$ is the number of bonds and $\tau_{ij}$ $(=\pm 1)$ denotes the sign of $J_{ij}$. It is reasonable to expect that, if the original system with parameters $(K, K_p)$ is on the crossover line (which corresponds to $K=K_p$), the transformed system with $(K'=R(K), K_p'=R(K_p))$ is on the same line corresponding to $K'=K_p'$. As a consequence, the crossover line is an invariant line, and therefore we can estimate the exponent $\nu$ for the crossover line as well as at $p=0.83$ and 0.90 by eqs. (2.13) and (2.14).

To check if the simulated system is in equilibrium, we use the exact results on the crossover line, $\exp(-2\beta J)=(1-p)/p$. The following relations can be proved for the energy and the correlation function on the line:[9]

$$[\langle E\rangle]_c=-\frac{1}{2}zN(2p-1), \tag{2.23}$$

$$[\langle S_iS_j\rangle]_c=[\langle S_iS_j\rangle^2]_c, \tag{2.24}$$

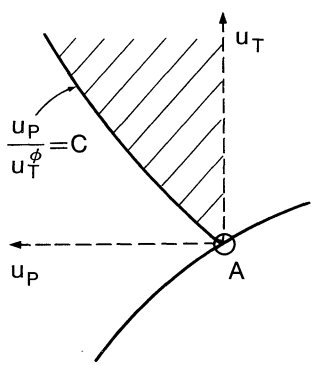


Fig. 1. Scaling fields $u_T$, $u_p$ and the invariant line defined by $u_p/u_T^{\phi}=c$. The critical line is shown by a bold line and we consider a point A on it. In the shaded region conventional exponents such as $\nu$, $\gamma$ and $\beta$ can be related with the exponents $y_T$ and $y_H$.

where $\langle \cdots \rangle$ denotes the thermal average for a given bond configuration and $[\cdots]_c$ represents the average over bond configurations. $z$ is the number of nearest neighbors, and $N$ is the number of sites. By dividing the whole MC step into 100 intervals and calculating the energy, magnetization and correlation functions in each interval, we regard the system as being in equilibrium after the errors in numerical estimations of eqs. (2.23) and (2.24) in a single interval become smaller than 1%. Although we simulate only one sample of bond configuration for a given set of $p$ and $T$, the system is so large ($N=32^3$ or $N=64^2$) that the sampling errors of thermodynamic quantities due to a specific bond configuration are negligible (of order $1/\sqrt{N}$). This assertion is justified also by observing that the resulting data of MCRG form smooth lines in the scaling analysis as shown in §3 in spite of the fact that we took different samples for different set of parameters $p$ and $T$. Therefore we use $\langle E \rangle$ in eq. (2.23) in our numerical calculations instead of $[\langle E \rangle]_c$, and $\Sigma_{(ij)} \langle S_i S_j \rangle$, $\Sigma_{(ij)} \langle S_i S_j \rangle^2$ in eq. (2.24) instead of $[\langle S_i S_j \rangle]_c$, $[\langle S_i S_j \rangle^2]_c$, where the sum is over $N$ randomly chosen pairs of sites. For $p=0.83$ and 0.90, since the exact relations such as eqs. (2.23) and (2.24) are not available, the check of equilibrium has been done only by dividing the whole step into 100 intervals as before and observing the stability of physical quantities.

## §3. Results and Discussion

The MCRG results on $-\Delta_{nm}(T)$ and $\Gamma_{nm}(T)$ as functions of $T$ in the case of the simple cubic lattice are plotted in Figs. 2 and 3 for the crossover line, in Figs. 4 and 5 for $p=0.83$, and in Figs. 6 and 7 for $p=0.90$. The number of steps of renormalization is indicated by $[n, m]$. We remark here that, on the crossover line, two data points (out of ten) in the low temperature region are omitted from Figs. 2 and 3 because they did not reach equilibrium even after $1.6\times10^6$ MCS/spin. For fixed $p$, the number of MCS ($=4\times10^5$/spin) is much smaller than on the crossover line because of the short relaxation time to equilibrium on these lines with fixed $p$, which probably comes from the fact that the SG phase is located farther away in the $p$-$T$ plane for those values of $p$.

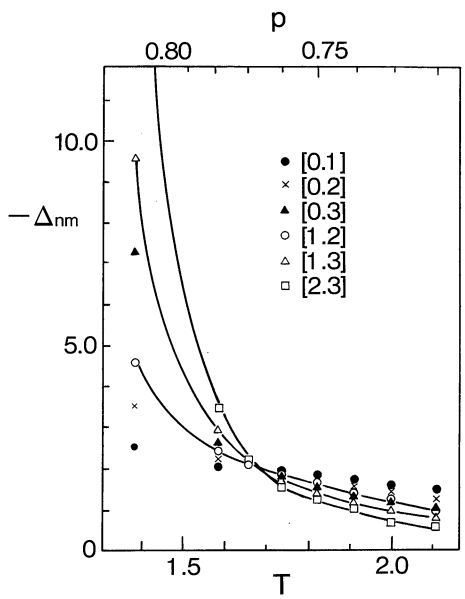


Fig. 2. Plot of $-\Delta_{nm}(T)$ vs $T$ (or $p$) of various $[n, m]$ for the crossover line on the simple cubic lattice. The interpolated curves for [1, 2], [1, 3] and [2, 3] intersect each other at $T=1.68$ ($p=0.767$) and $-\Delta_{nm}=1.97$.

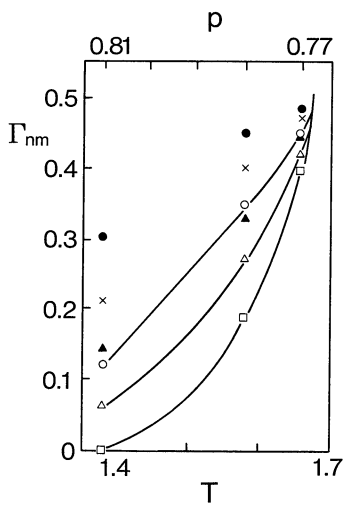


Fig. 3. Plot of $\Gamma_{nm}(T)$ vs $T$ for the crossover line on the simple cubic lattice. The symbols are the same as in Fig. 2.

Following ref. 12, the data for [1, 2] [1, 3] and [2, 3] are used to estimate $T_c$ since the original system (corresponding to $n=0$) is far from the fixed point even at its criticality ($T=T_c$). In order to estimate the precise value of $T_c$, we interpolate four data points nearest

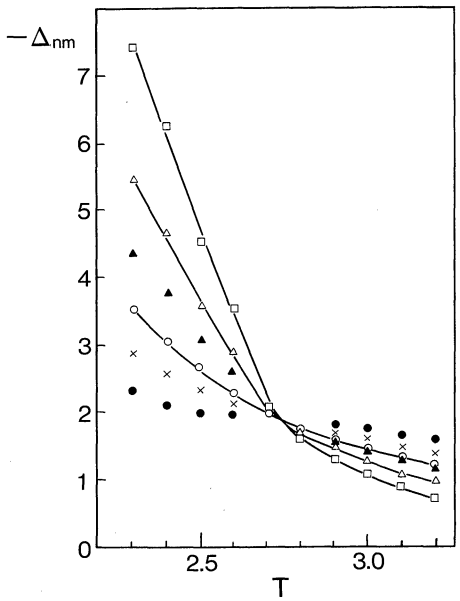


Fig. 4. Plot of $-\Delta_{nm}(T)$ vs $T$ for $p=0.83$ on the simple cubic lattice.

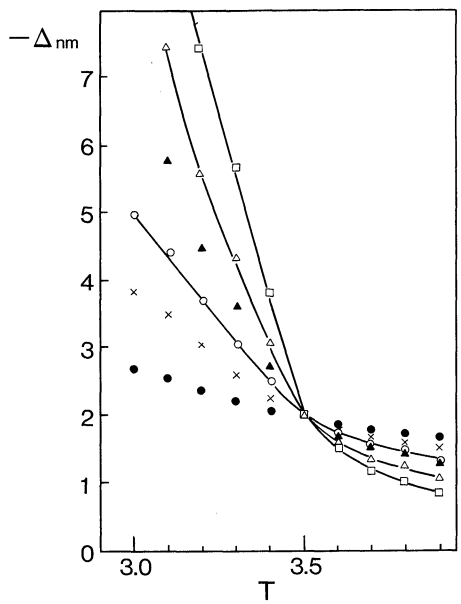


Fig. 6. Plot of $-\Delta_{nm}(T)$ vs $T$ for $p=0.90$ on the simple cubic lattice.

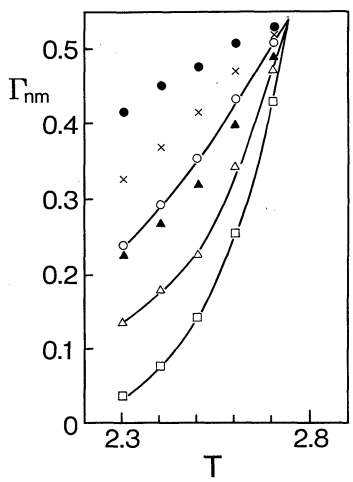


Fig. 5. Plot of $\Gamma_{nm}(T)$ vs $T$ for $p=0.83$ on the simple cubic lattice.

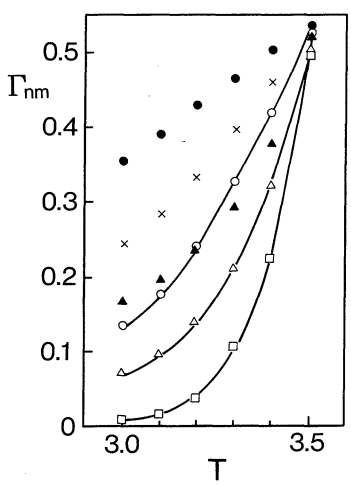


Fig. 7. Plot of $\Gamma_{nm}(T)$ vs $T$ for $p=0.90$ on the simple cubic lattice.

to $T_c$ by a cubic curve. In any one of Figs. 2–7, the interpolation curves for [1, 2], [1, 3] and [2, 3] intersect each other at almost the same point. According to eqs. (2.9) and (2.10), $T_c$, $\gamma/\nu$ and $\beta/\nu$ are obtained from these crossing points of interpolation curves. The obtained values are listed in Table I together with those for other $p$ and/or by other methods, for reference. Errors are estimated as the difference between the crossing point and the nearest data point (which should give an overestimation). These results on the critical points are plotted on the $p$-$T$ phase diagram in Fig. 8. Next we use eq. (2.14) to determine $\nu$. $[\partial\Delta_{nm}/\partial T]_{T=T_c}$ has been obtained from the slope of the interpolation curves which were used to determine the critical point and the exponent. The results are in Table I. Errors in $\nu$ are estimated from the uncertainty in $T_c$ by using the upper and lower limits of $T_c$ in eq. (2.14). To check the accuracy of this method to evaluate $\nu$, we have applied it to the non-

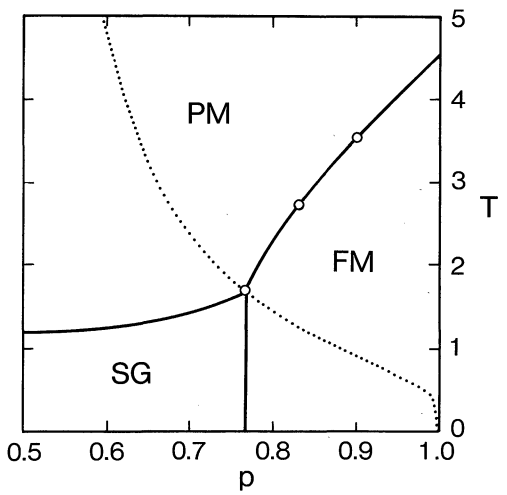


Fig. 8. The phase diagram of the simple cubic lattice. PM and FM denote the paramagnetic and ferromagnetic phases, respectively. Critical points ($p=0.900$, $T=3.51$) and (0.830, 2.73), and tricritical point (0.767, 1.68) are obtained by the present calculations (shown by white circles). The straight line from (0.767, 1.68) to (0.767, 0) separates the SG and FM phases as discussed in ref. 11. The non-random critical point is at (0, 4.51)[15] and the symmetric case has (0.5, 1.2).[5-7,17] The dotted line indicates the crossover line, $\exp(-2\beta J)=(1-p)/p$.

random system ($p=1$). First, we took data points very close to $T_c$, and obtained $\nu=0.64\pm0.02$ while the precise value is believed to be $\nu=0.630\pm0.0015$.[15,16] Next we took data points for each of which the value of $\Delta_{nm}(T)+\gamma/\nu$ is comparable to that of the present calculation on the random system. The result was $\nu=0.65\pm0.04$. Therefore our estimation of $\nu$ for the crossover line in the random system is reliable within the errors indicated in Table I.

Since the resulting values of $\nu$ are close to that of the non-random system except that at the tricritical point, universality between the non-random and random systems is expected to hold on the ferromagnetic critical line except at the tricritical point. What is more, as seen in Table I, the values of $\gamma/\nu$ and $\beta/\nu$ along the ferromagnetic critical line (*including* the tricritical point) coincide with those of the non-random system within the statistical uncertainty. Consequently weak universality is likely to hold along the phase boundary including both end points.

For the square lattice, the MCRG results of $-\Delta_{nm}(T)$ as a function of $T$ on the crossover line is plotted in Fig. 9, and $T_c$ and $\gamma/\nu$ are estimated by the same way as in $3d$ with exception of the interpolation which was made by the straight lines among data points. The obtained values are given in Table II. Weak universality seems to hold along phase boundary between the non-random case and the crossover line. Note that, as seen in Fig. 9, the MCRG data on the square lattice do not fall on a very smooth line as compared with those in $3d$. We have thus avoided to evaluate $\nu$ and $\beta/\nu$ with confidence. A possible reason for this uncertainty is that the block size, $b^d$, of an RG transformation is too small (4 for the square lattice whereas 8 for the simple cubic lattice) to assure stable results. The critical point thus

Table I. The estimated values of $T_c(p_c)$, $\gamma/\nu$, $\beta/\nu$, and $\nu$ for the simple cubic lattice. Those from various references are also listed. On $p=0.5$ the paramagnetic-SG phase transition occurs. Errors for $T_c(p_c)$ are estimated as the difference between the crossing point and the nearest data point in Figs. 2–7, those for $\nu$ are done from the uncertainty in $T_c$, and those for $\gamma/\nu$ and $\beta/\nu$ are given from the uncertainty in the corresponding calculations of the non-random case. All these methods should give an overestimation of errors.

| Line | $T_c$ | $\gamma/\nu$ | $\beta/\nu$ | $\nu$ | |
|---|---|---|---|---|---|
| Crossover line | 1.68 ±0.025 | 1.97±0.1 | 0.51±0.05 | 0.51±0.06 | |
| | ($p_c$= 0.767±0.004) | | | | |
| $p=0.83$ | 2.73 ±0.03 | 1.91±0.1 | 0.54±0.05 | 0.63±0.05 | |
| $p=0.90$ | 3.51 ±0.01 | 1.90±0.1 | 0.55±0.05 | 0.66±0.05 | |
| Crossover line | 1.8 ($p_c=0.75$) | | | | (ref. 8) |
| | 1.8 | | | | (ref. 14) |
| $p=1$ | 4.5115 | | | 0.629 | (ref. 15) |
| | | 1.968 | (0.516) | 0.630 | (ref. 16) |
| $p=0.5$ | 1.2 | | | | (ref. 5, 6) |
| | 1.175±0.025 | | | | (ref. 7, 17) |

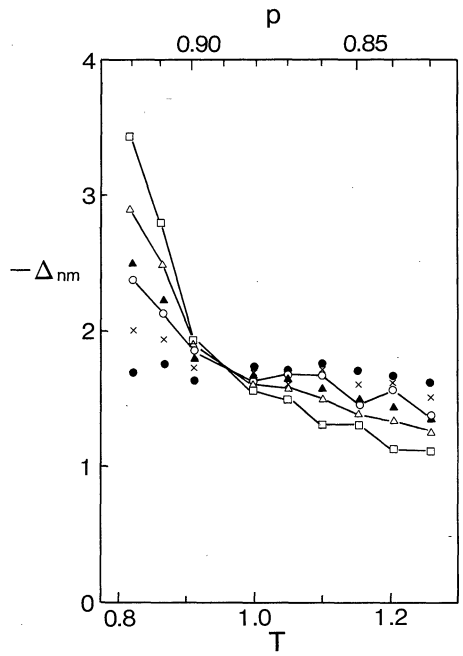


Fig. 9. Plot of $-\Delta_{nm}(T)$ vs $T$ for the crossover line on the square lattice.

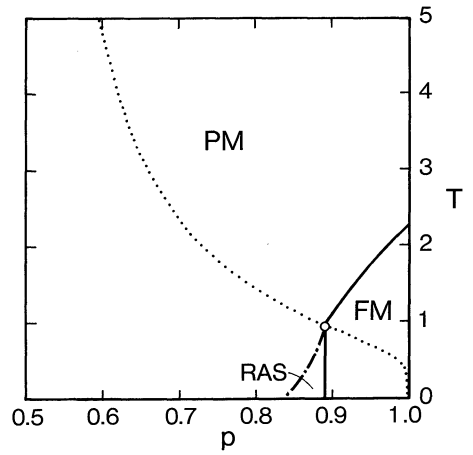


Fig. 10. The phase diagram of the square lattice. The critical point ($p=0.89$, $T=0.96$) is obtained by the present simulation. The straight line from this point to (0.89, 0) is drawn following ref. 11. The dotted line indicates the crossover line, $\exp(-2\beta J)=(1-p)/p$. The possible existence of a random antiphase state (RAS)[18] is noted.

obtained is plotted in the $p$-$T$ phase diagram in Fig. 10.

## §4. Summary and Remark

We have performed Monte Carlo simulations for the $\pm J$ Ising model on the simple cubic and the square lattices with asymmetric probability of ferromagnetic and antiferromagnetic bonds to estimate $T_c$, $\gamma/\nu$, $\beta/\nu$ and $\nu$ at various points in the phase diagram. The obtained values and resulting phase diagrams are listed in Table I and Fig. 8 for the simple cubic lattice and in Table II and Fig. 10 for the square lattice. These are much more accurate than those estimated so far (see references in Tables). In the simple cubic lattice, we provided evidence to believe that universality of ferromagnetic critical exponents hold except at the tricritical point, while weak universality does hold even at the tricritical point in the $p$-$T$ plane. In the square lattice, weak universality seems to hold along phase boundary between the non-random case and the crossover line. Concerning the crossover effect at the tricritical point in the simple cubic lattice, we note that the conventional tricritical scaling theory[13] cannot be readily applied to the present problem because the energy is an analytic function of $p$ on the crossover line,[9] which places an extra condition to be satisfied by the scaling functions. This problem should be solved in future.

Table II. The estimated values of $p_c(T_c)$ and $\gamma/\nu$ for the square lattice. Those from various references are also listed. Errors are estimated similarly to Table I.

| Line | $p_c$ | $\gamma/\nu$ | |
|---|---|---|---|
| Crossover line | $0.89\pm0.01$ ($T_c=0.96\pm0.05$) | $1.75\pm0.1$ | |
| Crossover line | 0.88 | | (ref. 19) |
| | 0.9 | | (ref. 20) |
| | $0.89\pm0.02$ | | (ref. 21) |
| $p=1$ | ($T_c=2.269$) | 1.75 | (ref. 22) |

## Acknowledgment

We gratefully acknowledge useful discussions with Seiji Miyashita, Hiroshi Takano, Macoto Kikuchi and Yutaka Okabe.

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