Journal of the Physical Society of Japan
Vol. 56, No. 9, September, 1987, pp. 3265-3269

Phase Diagram of the $\pm J$ Ising Model in Two Dimensions

Yukiyasu OZEKI and Hidetoshi NISHIMORI

Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152

(Received April 10, 1987)

The $\pm J$ Ising model with asymmetric probability weight of $\pm$bonds in two dimensions is studied in the p-T plane by the numerical transfer-matrix method. We calculate finite-width susceptibilities and finite-width correlation lengths of long strips of various sizes up to 14×10^5. Finite-size scaling analysis suggests that the random antiphase state exists adjacent to the ferromagnetic phase. The boundary separating these phases is vertical (parallel to the temperature axis) in the phase diagram. The weak universality seems to hold along the ferromagnetic critical line.

§1. Introduction

The finite dimensional spin glass (SG) models are of active current interest. The symmetric $\pm J$ Ising model is now believed to have its lower critical dimension d_{lc} in the range $2<d_{lc}\leq 3$, from careful Monte Carlo simulations[1-3] and long series of high temperature expansions.[4] As for the asymmetric $\pm J$ Ising model, we carried out Monte Carlo renormalization group (MCRG) calculations in our previous paper[5] on the simple cubic lattice and obtained the phase diagram and critical exponents. We also pointed out that the weak universality[6] of the ferromagnetic critical exponents along the phase boundary between ferromagnetic and paramagnetic states seems to hold for this system. However a similar MCRG procedure did not give satisfactory results in two dimensions mainly because of large statistical errors.

In this two-dimensional system, Morgenstern and Binder[7,8] obtained a rough phase diagram in the p-T plane by the transfer-matrix method. Nishimori[9] argued that the ferromagnetic critical line is vertical below the crossover line, defined by $\exp(-2\beta J)=(1-p)/p$, in the p-T plane, implying the absence of a reentrant transition from ferromagnetic to non-ferromagnetic phases. Maynard and Rammal[10] proposed that a highly correlated state, called the random antiphase state (RAS), exists near the ferromagnetic phase in the low temperature region, but Morgenstern[11] claimed that transfer-matrix calculation indicates the absence of RAS.

In the present paper, we exploit the numerical transfer-matrix method, orders of magnitude larger in scale than used previously, to resolve these problems in the two-dimensional $\pm J$ Ising model. We obtain the phase diagram accurately, and in particular find out a vertical critical line below the crossover line. Thus a reentrant transition does not exist in this system. We also provide strong evidence for the existence of RAS. In the next section, we explain the method of our calculation and its analysis. In §3, numerical results are presented as well as discussion. A few remarks are made in the last section.

§2. Method of Calculation

We treat the $\pm J$ Ising model with asymmetric probability weight on the square lattice with cylindrical boundary condition. Our model Hamiltonian is

$$\mathcal{H}=-\sum J_{ij}S_iS_j \qquad (S_i=\pm 1), \tag{2.1}$$

where the interaction J_{ij} is only in the nearest-neighbor bonds and is randomly distributed independently at each bond with the probability:

$$P(J_{ij})=p\delta(J_{ij}-J)+(1-p)\delta(J_{ij}+J). \tag{2.2}$$

We measure the temperature in units of J/k_B hereafter.

We consider systems of long strip of width L lattice spacings and calculate the "numerically exact" partition function by the transfer-matrix method.[7] Taking sufficiently long strips eliminates the necessity to average over various bond configurations. To obtain the critical line in the p-T phase diagram, we analyze the finite-width susceptibility $\chi_L(T, p)$ of the $L\times\infty$ system and the finite-width correlation length $\xi_L(T, p)$ defined by

$$|\langle S_0 S_R\rangle_L| \sim \exp(-\xi_L/R), \tag{2.3}$$

along the length of the strip.

The susceptibility $\chi_L(T, p)$ is obtained by numerical differentiation of the free energy density:

$$f_L(H)-f_L(0)=-\chi_L H^2+O(H^4). \tag{2.4}$$

Calculations were carried out at about 60 points in the p-T plane for lattices of $L\times 10^5$ with $L=6$, 8, 10, 12. These strips are long enough to obtain the numerical value of χ_L within the statistical error of 1%. The analysis follows the finite-size scaling hypothesis

$$\chi_L(u_T, u_p)=L^y g(L^{y_T}u_T, L^{y_p}u_p), \tag{2.5}$$

where u_T and u_p are the scaling fields corresponding to the temperature T and the probability p. The function g is the universal scaling function. The exponents y, y_T and y_p are written in terms of the ordinary critical exponents as γ/ν, $1/\nu$ and $1/\nu_p$, respectively. From eq. (2.5), we may assume

$$\chi_L(T, p)\propto L^{a(T,p)}, \tag{2.6}$$

in the close vicinity of a critical point ($u_T=u_p=0$). Then we can estimate $a(T, p)$ for each data point (T, p) from the L-dependence of χ_L. If the weak universality of the ferromagnetic transition is valid along the critical line extending from the non-random case ($p=1$), then γ/ν $(=\gamma_p/\nu_p)=1.75$ and the critical points can be obtained from the condition

$$a(T, p)=1.75, \tag{2.7}$$

as seen by comparison of eqs. (2.5) and (2.6). Actually the susceptibility does not follow the power-low, eq. (2.6), if the parameters (T, p) deviate from the critical values as one is convinced from eq. (2.5). The general behavior is schematically depicted in Fig. 1 in which linear fits of data points (circles) are shown in dashed lines. Clearly, even though the linear fit does not work away from criticality, this analysis correctly predicts the critical point.

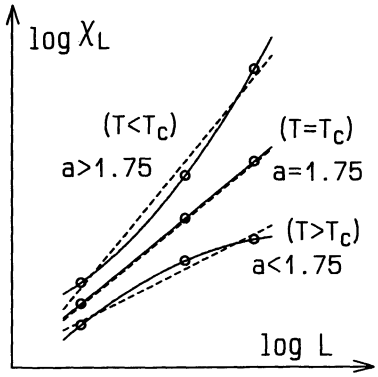

Fig. 1. The schematic plot of L-dependence of χ_L. Open circles denote given data. Full curves indicate smooth interpolation of data and dashed lines are the linear fits. It is noted that the slope of the line is 1.75 if it is at the criticality, less than 1.75 if above, and larger than 1.75 if below.

The method of calculations of ξ_L for the random systems has been described by Cheung and McMillan.[12] The ratio of largest two eigenvalues of the transfer matrix readily leads to the correlation length just as in the non-random case.* Calculations were carried out almost 140 points in the p-T plane for strips of $L\times 10^5$ with $L=6$, 8, 10, 12, 14. Similarly to the analysis of χ_L, we use the finite-size scaling hypothesis

$$\xi_L(u_T, u_p)=L\bar{g}(L^{y_T}u_T, L^{y_p}u_p), \tag{2.8}$$

and obtain $b(T, p)$ defined by

$$\xi_L(T, p)\propto L^{b(T,p)}. \tag{2.9}$$

The critical points are thus found from the condition

$$b(T, p)=1, \tag{2.10}$$

because only the linear power of L appears in eq. (2.8) if $u_T=u_p=0$. All computations were performed on HITAC S810/20 at the Com-

* We thank I. Morgenstern for drawing our attention to this fact.

puter Center of the University of Tokyo.

It should be remarked here that the criteria (2.7) and (2.10) do not always yield the same critical points. The susceptibility diverges only at the ferromagnetic critical points, while the correlation length defined by eq. (2.3) becomes singular at any phase transition where the order

$$\lim_{R\to\infty} |\langle S_0 S_R\rangle| \neq 0, \qquad (2.11)$$

builds up, as in SG or RAS. Therefore the paramagnetic-ferromagnetic transition is signaled by the both quantities χ and ξ, but the paramagnetic-RAS/SG change is accompanied only by the divergence of ξ. The boundary between the ferromagnetic and RAS/SG phases is characterized by the susceptibility divergence only.

§3. Results and Discussion

First we present the analysis of χ_L. Making use of eq. (2.6), we obtain $a(T,p)$ at 60 different points of (T,p) for $L=\{6, 8, 10\}$ and at 30 points for $L=\{8, 10, 12\}$. The latter large L's were used at closer points to the critical line. We then interpolate the obtained $a(T,p)$ as a function of T (or p) to determine the critical line by the condition eq. (2.7). The result is plotted as crosses in Fig. 2. The full curve represents the critical line obtained from smooth interpolation between crosses. The dashed line is the exact tangent line[13] of the critical line at the non-random critical point ($T=2.269\cdots$, $p=1$). Excellent agreement is apparently achieved. The critical point on the crossover line (dotted line in Fig. 1) obtained by the present calculations, ($T=0.961\pm0.009$, $p=0.889\pm0.002$), agrees with that by the MCRG,[5] $p=0.89\pm0.01$. As a consequence, the assumption of the weak universality (2.7) is valid along the critical line from the non-random critical point down to the crossover line. The vertical sector below the crossover line is seen to have a slight leftward curvature and this point will be discussed later.

Next we discuss the correlation length ξ_L. The results are plotted in Fig. 3. The analysis proceeded in the same way as in the case of χ_L. The full line represents the ferromagnetic critical line estimated from χ_L. Three different sets of L, $\{6, 8, 10\}$ (circles), $\{8, 10, 12\}$ (crosses) and $\{10, 12, 14\}$ (squares), are used to evaluate $b(T,p)$ in eq. (2.9). The critical points located between the non-random system ($p=1$) and the crossover line lie precisely on the full curve from χ_L. This fact endorses reliability of the present method to estimate the critical line in this random system. In order to clarify the behavior of the critical line

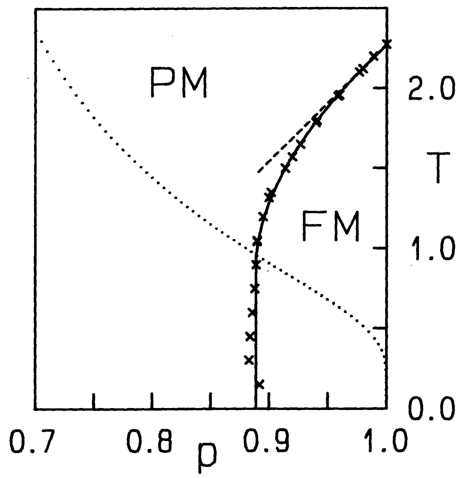

Fig. 2. The critical line (full curve) estimated by χ_L. Numerically estimated points are plotted by crosses. The exact tangent line of the critical line at the non-random critical point is shown by the dashed line. The crossover line, $\exp(-2\beta J)=(1-p)/p$, is also indicated as the dotted curve.

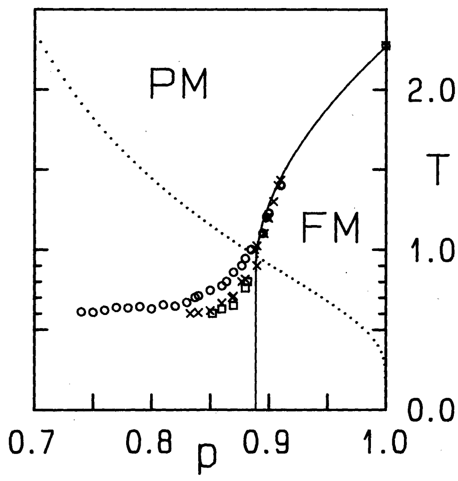

Fig. 3. The critical points estimated by ξ_L. Open circles, crosses and squares are the critical points from $L=\{6, 8, 10\}$, $\{8, 10, 12\}$ and $\{10, 12, 14\}$, respectively. The critical line estimated from χ_L is drawn as the full curve.

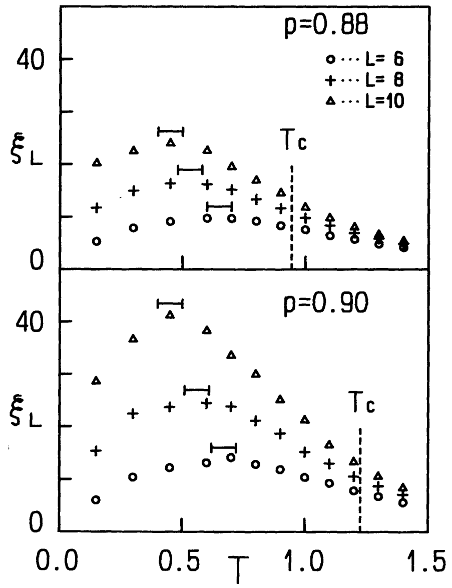

Fig. 4. Temperature dependence of ξ_L for $p=0.88$ and 0.90. Each bar indicates the range of existence of the peak of ξ_L. The dashed lines indicate the critical points suggested from the finite-size scaling analysis.

below the crossover line, we plot $\xi_L(T, p)$ as a function of T in Fig. 4 at $p=0.88$ and 0.90. The former p lies just out of the ferromagnetic region whereas the latter is just in it (see Fig. 2). Each bar indicates the range of existence of peak of ξ_L. The peak temperature decreases as L increases, and does not change drastically over $1 \geq p \geq 1/2$ for a given L. Therefore we may safely assume that the peak of ξ_L shifts monotonically toward zero temperature as L increases. This observation implies that the reentrant-like behavior (decrease of ξ as T decreases) does not exist in the infinite system. It is also apparent that the asymptotic behavior as $L \to \infty$ has not been reached in the temperature range below the peak. Consequently, we may use the finite-size scaling, eqs. (2.5) and (2.8), only in the range $T \gtrsim 0.7$ (above the peak temperature) for all p if $L \geq 6$. For the analysis of larger L we can use slightly wider range of temperature (i.e. $T \gtrsim 0.65$ for $L \gtrsim 8$). Four low-temperature points ($T=0.6$, 0.45, 0.3, 0.15) in Fig. 2 thus strongly reflect finite-size effects and should be ignored. The remaining critical points below the crossover line clearly form a vertical line as suggested in ref. 9. Returning to the result from the analysis of correlation length, Fig. 3, we can clearly observe the possible existence of RAS adjacent to the ferromagntic region even if we drop the data in the range $T<0.7$. The large deviation of open circles from crosses and squares comes from the fact that the former points were obtained from the data including size $L=6$, in which the peaks of ξ_L are located in high-temperature range. The final phase diagram is given in Fig. 5. The low-temperature boundary of RAS is drawn by simply extrapolating the line for $T \gtrsim 0.7$.

§4. Summary and Remark

We have investigated the asymmetric $\pm J$ Ising model in two dimensions by the transfer-matrix method, and obtained the phase diagram in Fig. 5. The weak universality[6] holds along the ferromagnetic critical line. A vertical ferromagnetic phase boundary predicted in ref. 9 is consistent with the weak universality along the boundary. We also obtained a strong indication from the analysis of the correlation length defined in eq. (2.3) that the random antiphase state (RAS) exists at finite temperatures adjacent to the ferromagnetic phase below the crossover line. Morgenstern,[11] on the other hand, denied the existence of RAS on the basis of transfer-matrix calculations of 20×12 and 12×16 systems. We point out here a few problems in his analysis. First, he extrapolates his correlation data to $T \to 0$ to investigate the ground

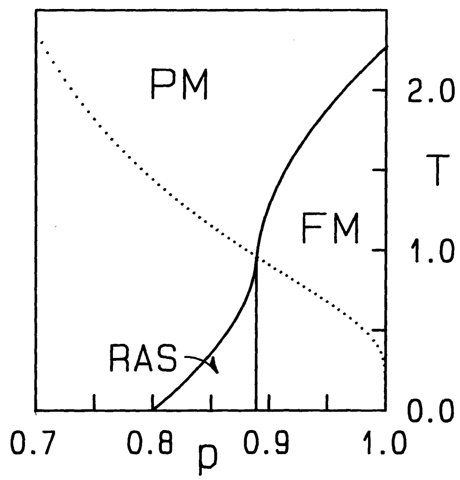

Fig. 5. Our result on the phase diagram of the asymmetric $\pm J$ Ising model in two dimensions.

state properties. However, his extrapolation to $T \to 0$ is dangerous in view of the erratic behavior of the correlation length as demonstrated in Fig. 4. What is more, he does not discuss the correlation length but the correlation function. Our direct evaluation of ξ is superior to his method, because we definitely observe the asymptotic long-distance behavior while he treated only limited length scale up to 12×20. If one takes a long strip, the correlation function will saturate to a finite value suggesting a finite RAS order even in the parameter region where he denies RAS.

References

1) R. N. Bhatt and A. P. Young: Phys. Rev. Lett. **54** (1985) 924.
2) A. T. Ogielski and I. Morgenstern: Phys. Rev. Lett. **54** (1985) 928.
3) A. T. Ogielski: Phys. Rev. B**32** (1985) 7384.
4) R. R. P. Singh and S. Chakravarty: Phys. Rev. Lett. **57** (1986) 245.
5) Y. Ozeki and H. Nishimori: J. Phys. Soc. Jpn. **56** (1987) 1568.
6) M. Suzuki: Prog. Theor. Phys. **51** (1974) 1992.
7) I. Morgenstern and K. Binder: Phys. Rev. Lett. **43** (1979) 1615.
8) I. Morgenstern and K. Binder: Phys. Rev. B**22** (1980) 288.
9) H. Nishimori: J. Phys. Soc. Jpn. **55** (1986) 3305.
10) R. Maynard and R. Rammal: J. Phys. Lett. (France) **43** (1982) L347.
11) I. Morgenstern: Phys. Rev. B**25** (1982) 6071.
12) H.-F. Cheung and W. L. McMillan: J. Phys. C**16** (1983) 7027.
13) E. Domany: J. Phys. C**12** (1979) L119.