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Nonlinear Master Equation Approach to Asymmetrical Neural Networks of the Hopfield-Hemmen Type

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Dynamical properties of the Hopfield-Hemmen-type neural networks with asymmetrical synaptic connections are studied using a nonlinear master equation which is obtained in the thermodynamic limit from the Glauber dynamics assumed for the networks. A self-consistent equation is derived for the time evolution of the overlaps of the instantaneous configuration with the built-in patterns of a finite number p. Two fundamental types of conditions for the occurrence of limit cycle-type oscillation of the pattern overlaps are presented for the case of $p=2$.

[Hopfield-Hemmen model, neural network, Glauber dynamics, nonlinear master equation, limit cycle, nonequilibrium phase transition]

§1. Introduction

The Hopfield model[1] for neural networks has attracted much attention in physical science in recent years. Stochastic versions of the original Hopfield model, which are of the Boltzmann machine type and are referred to as generalized Hopfield-Hemmen models,[2] have been extensively investigated from the viewpoint of the statistical mechanics of phase transitions in spin-glass-like systems. There, the synaptic connections between spinlike neurons are symmetric, and accordingly no oscillatory phenomena occur.

By contrast, there has been relatively little activity in investigations of neural network models having asymmetric connections,[3] although such connections are ubiquitous in real neurons of living systems. This is because an introduction of asymmetric connections between neurons into neural network systems, in general, defies a systematic analysis of the dynamical behavior of the system, owing to the fact that free energy functions no longer exist and hence a variety of complex nonlinear dynamical phenomena such as the limit cycle or chaos are expected to appear.

Our motivation for undertaking the present study of neural networks with asymmetric connections was twofold. First, by taking up as simple dynamical models as possible for the modeling of neural networks, we wanted to investigate the mechanisms underlying both the occurrence of limit cycle-type pattern evolution and the memory retrieval of temporal patterns.[3] The understanding of such temporally oscillating pattern generation seems to have potential applicability in the study of central pattern generators in some neural systems. It also seems to be of interest to see to what extent the appearance of spurious patterns can be suppressed on introduction of asymmetry in the synaptic connections. Second, we are interested in developing a theory of nonequilibrium phase transitions as a natural extension of thermodynamic phase transitions. We have so far been concerned with such phase transitions exhibited by stochastic systems of infinitely many coupled nonlinear oscillators on the basis of the nonlinear Fokker-Planck equation,[4] which is a kind of nonlinear master equation capable of displaying bifurcation phenomena. In the following, to study the dynamical behavior of the generalized Hopfield-Hemmen model with asymmetric connections, we take advantage of the nonlinear master equation associated with the Glauber dynamics for the systems.

§2. Nonlinear Master Equation and Self-Consistent Equation for the Pattern Overlaps

In the generalized Hopfield-Hemmen model, neural networks exhibiting features of learning and associative memory retrieval are described by a stochastic system of Ising spins with exchange interactions such that two levels of activity (firing and resting) of neurons are represented by bivariate spin variables $\{S_i\}$. Synaptic connections between neurons are modeled by exchange interactions J_{ij} with the Hebb learning rule taken into account. The synaptic connections J_{ij} which the j-th neuron makes with the i-th neuron are assumed to be given by

$$J_{ij}=\frac{1}{N}\sum_{\mu,\nu=1}^{p}\xi_i^{(\mu)}a_{\mu\nu}\xi_j^{(\nu)}, \tag{1}$$

with N representing the total number of neurons. The p sets of $\{\xi_i^{(\nu)}\}$ $(\nu=1,\cdots p)$ constitute memory patterns embedded in the neural networks, with each $\xi_i^{(\nu)}$ taking either $+1$ or -1. In the essence of the present theory, the $\xi_i^{(\nu)}$ need not be random and $\{\xi_i^{(\nu)}\}$ can be allowed to assume any arbitrarily chosen pattern. They are characterized by the rate of appearance of their component $r(\xi)$, which represents the ratio of the total number of neurons endowed with the vector of pattern components $\xi=(\xi^{(1)}, \xi^{(2)}, \cdots\xi^{(p)})$, which is defined in the p-dimensional hypercube H^p of ± 1 coordinates. When the $p\times p$ matrix $(a_{\mu\nu})$ representing the connection weights is symmetric, so become the connections J_{ij}. In our treatment, $(a_{\mu\nu})$ need not be symmetric for the purpose of dealing with the case of asymmetric connections.

The effective local field $h_i(=\sum_j J_{ij}S_j)$ to which the i-th neuron S_i is subjected is now given from eq. (1) as

$$h_i=\sum_{\mu,\nu}\xi_i^{(\mu)}a_{\mu\nu}g^{(\nu)}, \tag{2}$$

where we define the overlaps $g^{(\nu)}$ with the built-in patterns $\{\xi_i^{(\nu)}\}(\nu=1,\cdots p)$:

$$g^{(\nu)}=\frac{1}{N}\sum_{j=1}^{N}\xi_j^{(\nu)}S_j. \tag{3}$$

Incorporating this effective field into a transition rate, we consider Glauber dynamics[5] as governing the time evolution of the microscopic state $\{S_i\}$ of the neural networks. To be specific, if we assume the transition rate $w(S_i\to -S_i)$, as usual, to be $w(S_i\to -S_i)=1/2(1-S_i\tanh\beta h_i)$, the N-body probability distribution $P(S_1\cdots S_N;\, t)$ reads

$$\frac{\partial P(S_1\cdots S_N;\, t)}{\partial t}=-\sum_i w(S_i\to -S_i)P(S_1\cdots S_i\cdots S_N;\, t)+\sum_i w(-S_i\to S_i)P(S_1\cdots -S_i\cdots S_N;\, t). \tag{4}$$

Although in thermodynamic spin systems β represents inverse temperature, it will lose its meaning but represents a measure of inverse magnitude of external noise or perturbations in the case of neural networks, especially with general types of couplings. We might, however, refer to $1/\beta$ as "temperature" $(1/\beta=T)$ for the sake of convenience. Since p patterns are embedded for memories in the networks through the connections, we divide the system of N neurons into 2^p sublattices according to the vector of the built-in pattern components $\xi=(\xi^{(1)},\cdots\xi^{(p)})$ with which each neuron is endowed.

We now define, in the thermodynamic limit $N\to\infty$, the empirical probability $p(t|\xi)$ for each sublattice having ξ such that it represents the probability of finding $+1$ spin at time t in that sublattice. Then it is easy to see from the above Glauber dynamics that the time evolution of $p(t|\xi)$ obeys the following nonlinear master equation

$$\frac{d}{dt}p(t|\xi)=-\frac{1}{2}\{1-\tanh\beta h(\xi)\}p(t|\xi)+\frac{1}{2}\{1+\tanh\beta h(\xi)\}\{1-p(t|\xi)\}$$
$$=-p(t|\xi)+\frac{1}{2}\{1+\tanh\beta h(\xi)\}, \tag{5}$$

where $h(\xi)=\sum_{\mu,\nu}\xi^{(\mu)}a_{\mu\nu}g^{(\nu)}$. The overlap $g^{(\nu)}$ can be expressed in terms of the $p(t\,|\,\xi)$ as

$$g^{(\nu)}=\sum_{\xi\in H^p} r(\xi)\xi^{(\nu)}[p(t\,|\,\xi)-\{1-p(t\,|\,\xi)\}]$$
$$=\sum_{\xi\in H^p} r(\xi)\xi^{(\nu)}\{2p(t\,|\,\xi)-1\}. \qquad (6)$$

Then eq. (5) turns out to constitute a self-consistent equation determining the time evolution of the 2^p empirical probabilities. This nonlinear master equation is akin to the nonlinear Fokker-Planck equation obtained for systems of infinitely many coupled Langevin equations.[4] Differentiating eq. (6) with respect to t and substituting eq. (5), one obtains another self-consistent equation describing the time evolution of the overlaps in a p-dimensional phase space:

$$\frac{\mathrm{d}}{\mathrm{d}t}g^{(\nu)}=-g^{(\nu)}+\sum_{\xi\in H^p} r(\xi)\xi^{(\nu)} \times\tanh\Big(\beta\sum_{\mu,k}\xi^{(\mu)}a_{\mu k}g^{(k)}\Big),$$
$$\nu=1,\cdots p. \qquad (7)$$

Since the reduction of the number of variables from 2^p to p is remarkable in the analysis of dynamical systems of differential equations, eq. (7) turns out to play an important role in examining the dynamical behavior of the pattern evolution of the present systems, as shown below.

§3. Conditions for the Occurrence of Limit Cycle in the Case of $p=2$

To investigate the effect of asymmetry in the synaptic connections on the dynamical properties of the networks, we deal with the case where the networks have two built-in patterns ($p=2$). We assume here the absence of correlations between the patterns for simplicity and we set $r(1,1)=r_1r_2$, $r(1,-1)=r_1(1-r_2)$, $r(-1,1)=(1-r_1)r_2$, $r(-1,-1)=(1-r_1)(1-r_2)$, where r_ν ($\nu=1,2$) denotes the rate of appearance of $\xi^{(\nu)}=+1$ with respect to the ν-th pattern.

Since $g^{(1)}=g^{(2)}=0$ is observed to satisfy eq. (7) with $p=2$, conducting a linear stability analysis around this trivial solution yields the condition for Hopf bifurcation to occur:

$$4\Delta^2(1-v^2)>\{2b+v(a_{11}+a_{22})\}^2 +(a_{11}-a_{22})^2(1-v^2), \qquad (8)$$

with criticality given by $\beta_c(a_{11}+a_{22}+2bv)=2$ ($\beta_c>0$). Here, $\Delta=(a_{12}-a_{21})/2$ is a measure of the degree of asymmetry of the matrix $\{a_{\mu\nu}\}$, $b=(a_{12}+a_{21})/2$, and $v=(2r_1-1)(2r_2-1)$. Inequality (8) and stability condition for the Hopf bifurcation constitute conditions for the occurrence of a stable limit cycle (type 1 conditions). When $v=0$, the stability condition takes a simple form:

$$a_{11}+a_{22}>b(a_{11}-a_{22})/\Delta. \qquad (9)$$

We see under the type 1 conditions that while the paramagnetic state $(g^{(1)},g^{(2)})=0$ is stable at higher "temperatures" ($\beta\leqq\beta_c$), a limit cycle solution bifurcating at $\beta=\beta_c$ becomes stable in a certain region below the critical "temperature" ($\beta>\beta_c$). Figure 1 displays an example of such a limit cycle. The occurrence of Hopf bifurcation represents a characteristic feature of nonequilibrium phase transitions in stochastic systems without detailed balance.[4]

The persistency at lower temperatures up to $T=0$ ($\beta=\infty$) of the limit cycle solution obtained above depends on whether the parameters involved satisfy another type of condition for the occurrence of a limit cycle at $T=0$. Noting that $\tanh\beta(\,\bullet\,)\to\mathrm{sgn}\,(\,\bullet\,)$ as $\beta\to\infty$, eq. (7) in the zero-temperature limit reads

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}g^{(1)}=-g^{(1)}+\{r_1r_2+(1-r_1)(1-r_2)\}\,\mathrm{sgn}\,(y^{(1)})+\{r_1(1-r_2)+(1-r_1)r_2\}\,\mathrm{sgn}\,(y^{(2)})\\[2mm] \dfrac{\mathrm{d}}{\mathrm{d}t}g^{(2)}=-g^{(2)}+\{r_1r_2+(1-r_1)(1-r_2)\}\,\mathrm{sgn}\,(y^{(1)})-\{r_1(1-r_2)+(1-r_1)r_2\}\,\mathrm{sgn}\,(y^{(2)})\end{cases}, \qquad (10)$$

with $y^{(1)}=(a_{11}+a_{21})g^{(1)}+(a_{12}+a_{22})g^{(2)}$, $y^{(2)}=(a_{11}-a_{21})g^{(1)}+(a_{12}-a_{22})g^{(2)}$. A formal equilibrium solution C^{eq} is easily obtained by setting $\dot g^{(1)}=\dot g^{(2)}=0$ for each of the following four regions separated by the two straight lines in the $g^{(1)}-g^{(2)}$ plane:

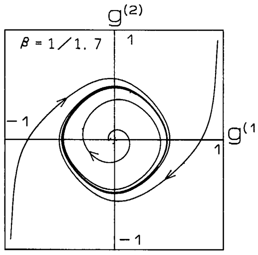

Fig. 1.

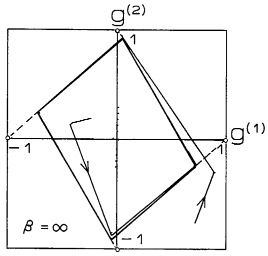

Fig. 2.

Fig. 1. A flow trajectory of eq. (7) settling into a limit cycle at $\beta=1/1.7$. The parameters are $a_{11}=2$, $a_{12}=1$, $a_{21}=-1$, $a_{22}=2$, $r_1=r_2=1/2$, which give a Hopf bifurcation point of $\beta_c=1/2$ and do not satisfy type 2 conditions.

Fig. 2. An example of retrieval of temporal pattern sequence. Shown is a flow trajectory of eq. (7) exhibiting limit cycle behavior at $\beta=\infty$ ($T=0$). The parameters are $a_{11}=7.6$, $a_{12}=-1$, $a_{21}=8$, $a_{22}=0.2$, $r_1=r_2=1/2$, which satisfy type 2 conditions alone, excluding the Hopf bifurcation of eq. (8). The four seeming equilibrium points, (1, 0), (0, 1), (−1, 0), and (0, −1), which correspond to the two embedded patterns and their reverses, are designated by circles (○). Each linear trajectory in the respective region separated by the two lines (dotted lines) has its own target of the seeming equilibrium point associated with the region. The embedded patterns are observed to be approximately retrieved successively: $\{\xi_i^{(1)}\}\to\{\xi_i^{(2)}\}\to\{-\xi_i^{(1)}\}\to\{-\xi_i^{(2)}\}\to\{\xi_i^{(1)}\}$.

$$
\begin{aligned}
C_{\mathrm{I}}^{\mathrm{eq}}&=(1,\ v) && \text{for } [\mathrm{I}]=\{y^{(1)}>0,\ y^{(2)}>0\}\\
C_{\mathrm{II}}^{\mathrm{eq}}&=(v,\ 1) && \text{for } [\mathrm{II}]=\{y^{(1)}>0,\ y^{(2)}<0\}\\
C_{\mathrm{III}}^{\mathrm{eq}}&=(-1,\ -v) && \text{for } [\mathrm{III}]=\{y^{(1)}<0,\ y^{(2)}<0\}\\
C_{\mathrm{IV}}^{\mathrm{eq}}&=(-v,\ -1) && \text{for } [\mathrm{IV}]=\{y^{(1)}<0,\ y^{(2)}>0\}.
\end{aligned} \tag{11}
$$

The above formal equilibrium points are actually realized only if they belong to their associated regions. When none of the above fixed points belong to their own regions, no equilibrium states representing particular built-in patterns or their reverses $\{-\xi_i^{(\mu)}\}$ are allowed at $\beta=\infty$. In particular, when $C_{\mathrm{I}}^{\mathrm{eq}}\in[\mathrm{II}]$, $C_{\mathrm{II}}^{\mathrm{eq}}\in[\mathrm{III}]$ or when $C_{\mathrm{I}}^{\mathrm{eq}}\in[\mathrm{IV}]$, $C_{\mathrm{II}}^{\mathrm{eq}}\in[\mathrm{I}]$, flows governed by eq. (10), whose trajectory is represented by piecewise straight lines, can easily be shown to circulate around the origin, and under certain existence and stability conditions they settle into a stable limit cycle, an example of which is given in Fig. 2. We have confirmed that the above two conditions, namely, of circulation and of existence and stability, constitute conditions for the occurrence of a stable limit cycle in the limit $\beta\to\infty$ (type 2 conditions). Since explicit expressions for such conditions in the general case of arbitrary values of r_v become a bit lengthy, we describe only the result for the case of $r_1=1/2$ or $r_2=1/2$ ($v=0$). Then the conditions (type 2) can be summarized in a compact form as

$$-1<\frac{a_{22}}{a_{12}}<\frac{a_{11}}{a_{21}}<1,\quad a_{21}>0,\ a_{12}<0,$$

or

$$-1<\frac{a_{11}}{a_{21}}<\frac{a_{22}}{a_{12}}<1,\quad a_{21}<0,\quad a_{12}>0. \tag{12}$$

When both of the conditions of type 1 and 2 are satisfied, a limit cycle solution remains in existence for $\beta>\beta_c$, until it vanishes in amplitude via Hopf bifurcation at $\beta=\beta_c$. In the case where only the type 1 conditions are satisfied, the limit cycle which emerged via Hopf bifurcation disappears, as β is increased, at a certain value of $\beta(\infty\geqq\beta>\beta_c)$, giving way to equilibrium solutions. In the case where only type 2 conditions are met, the limit cycle suddenly disappears at a certain value of β, as β is decreased from $\beta=\infty$, and depending on the values of parameters, it happens that a further decrease in β results in an occurrence of pitchfork bifurcation at $(g^{(1)}, g^{(2)})=0$. The above situations are again characteristic of the nonequilibrium phase transitions in the present stochastic systems. Details including a thorough analysis of the bifurcation scheme for eq. (7) will be published elsewhere.

After completion of this work, an analysis of the dynamical behavior of similar asymmetric networks which is less complete than ours appeared.[6]

References

1) J. J. Hopfield: Proc. Natl. Acad. Sci. USA **79** (1982) 2554; **B1** (1984) 3088.
2) J. L. Van Hemmen: Phys. Rev. Lett. **49** (1982) 409; D. J. Amit, H. Gutfreund and H. Sompolinsky: Phys. Rev. **A32** (1985) 1007.
3) H. Sompolinsky and I. Kanter: Phys. Rev. Lett. **57** (1986) 2861; S. Shinomoto: Biol. Cybern. **57** (1987) 197.
4) M. Shiino: Phys. Lett. **111A** (1985) 396; **112A** (1985) 302; Phys. Rev. **A36** (1987) 2393.
5) R. J. Glauber: J. Math. Phys. **4** (1963) 294; M. Suzuki and R. Kubo: J. Phys. Soc. Jpn. **24** (1968) 51.
6) A. C. C. Coolen and Th. W. Ruijgrok: Phys. Rev. **A38** (1988) 4253.