Mean-field Analysis of Quantum Annealing with XX-type Terms

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Y. Seki and H. Nishimori, Phys. Rev. E85, 051112 (2012)

B. Seoane and H. Nishimori, J. Phys. A45, 435301 (2012)

Y. Seki and H. Nishimori, J. Phys. A48, 335301 (2015)

See also T. Kadowaki and H. Nishimori, Phys. Rev. E58, 5355 (1998)

Problem

Find the ground state of Ising model starting from paramagnet

$$H = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p \quad (p = 3, 5, 7, \ldots)$$

$$\left|\uparrow\uparrow\uparrow\ldots\uparrow\right\rangle$$

Quantum annealing with transverse field

$$H(s) = -sN\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - (1-s)\sum_{i=1}^{N}\sigma_i^x$$

$$t = 0 : s = 0, H(0) = -\sum_i \sigma_i^x \quad \left|g\right\rangle = \left|\rightarrow\rightarrow\rightarrow\cdots\rightarrow\right\rangle = \underbrace{\left|\uparrow\uparrow\ldots\uparrow\right\rangle + \left|\uparrow\uparrow\ldots\downarrow\right\rangle\cdots + \left|\downarrow\downarrow\ldots\downarrow\right\rangle}_{2^N}$$

Quantum para

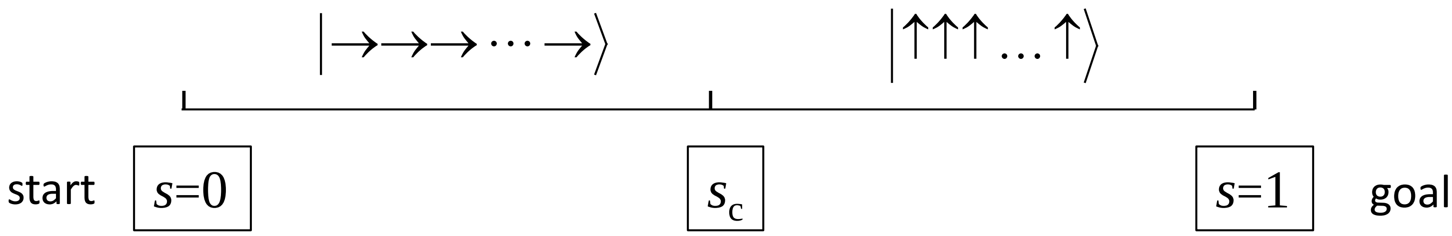
$$t = \tau : s = 1, H(1) = -N\left(\frac{1}{N}\sum_i \sigma_i^z\right)^p \quad \left|g\right\rangle = \left|\uparrow\uparrow\uparrow\cdots\uparrow\right\rangle$$

Ordered state

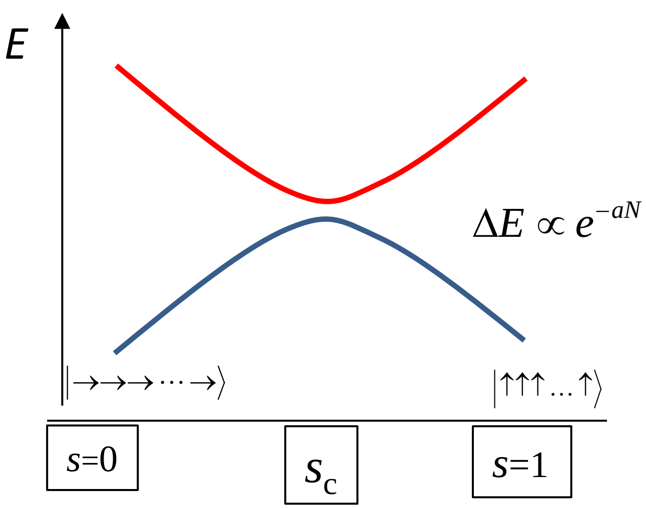
1st order transition at $s=s_c$

Jorg et al 2010

1st order quantum transition



Phase transition



1st order: exponentially long time; Hard to solve

$$\tau \propto \frac{1}{(\Delta E)^2} \propto e^{2aN}$$

reduction

2nd order: moderate time; Easy to solve

$$\tau \propto \frac{1}{(\Delta E)^2} \propto N^b << e^{2aN}$$

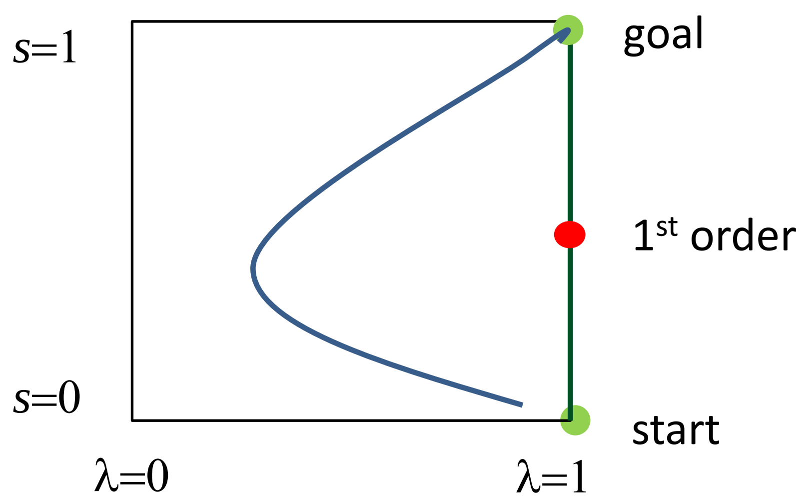
Solution: antiferromagnetic *XX* interaction

$$H(s,\lambda) = -s\left[\lambda N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - (1-\lambda)N\left(\frac{1}{N}\sum_{i}\sigma_i^x\right)^2\right] - (1-s)\sum_{i=1}^{N}\sigma_i^x \quad \begin{pmatrix} * & + & - & + \\ + & * & 0 & - \\ - & 0 & * & - \\ + & - & - & * \end{pmatrix}$$

Non-stoquastic

$$s=0: H(0,\lambda) = -\sum_i \sigma_i^x \quad \text{(start)}$$

$$s=\lambda=1: H(1,1) = -N\left(\frac{1}{N}\sum_i \sigma_i^z\right)^p \quad \text{(goal)}$$



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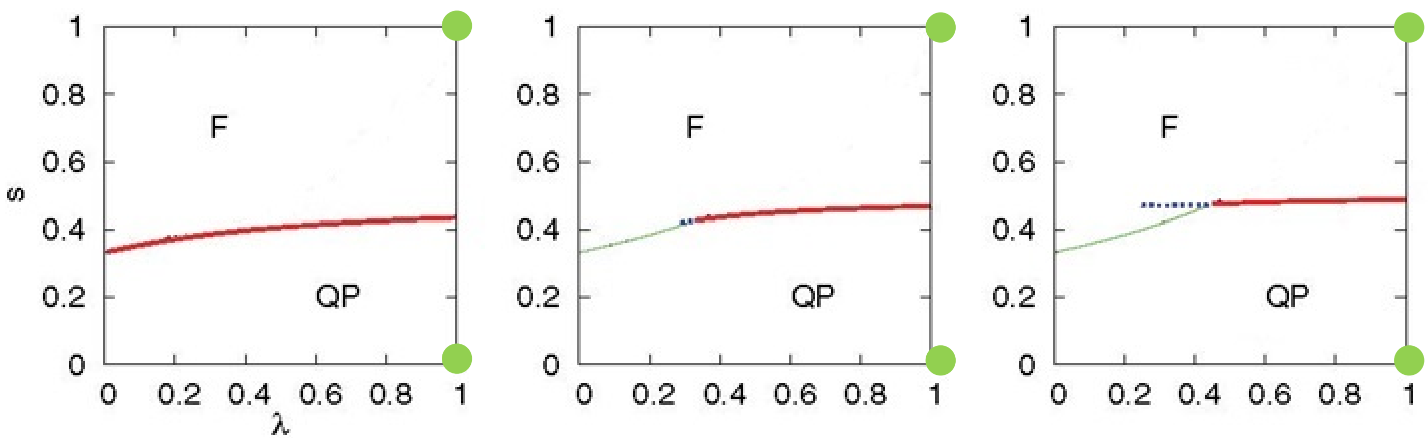
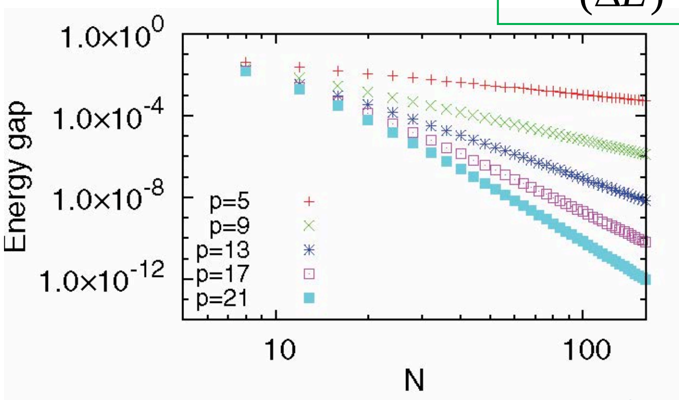


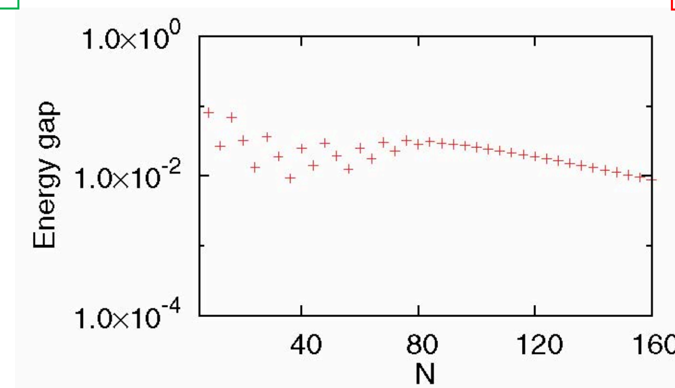
p=3 *p*=5 *p*=11

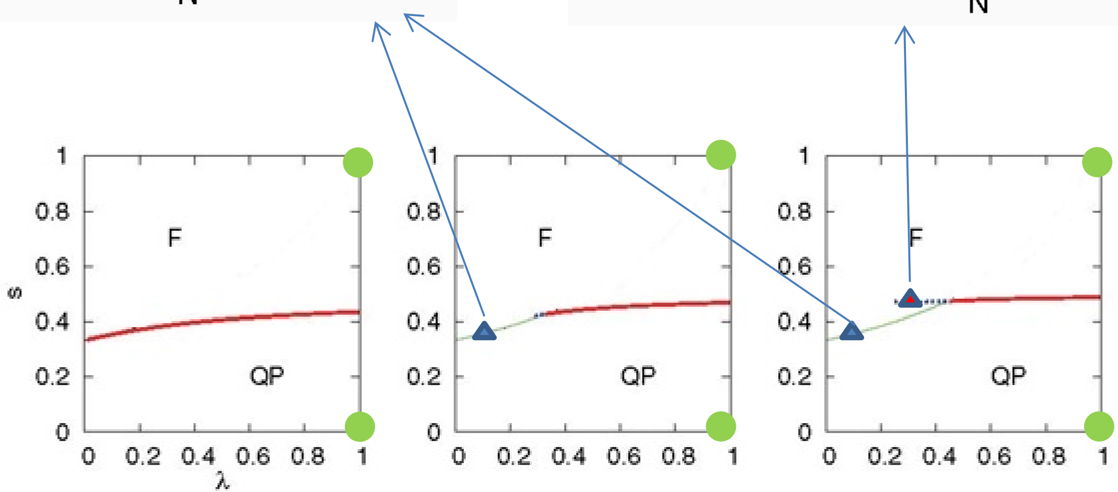
Exponential vs polynomial rate of gap closing

λ=0.1 polynomial $\tau \propto \frac{1}{(\Delta E)^2} \propto N^b$



λ=0.3, p=11 exponential $\tau \propto \frac{1}{(\Delta E)^2} \propto e^{2aN}$





p=3 p=5 p=11

Why is p=3 special?

Landau free energy

$$H = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p$$

$$p = 3$$

$$F(m) = am^2 + bm^3 + \cdots$$

$$p \geq 5$$

$$F(m) = am^2 + cm^4 + \cdots$$

Asymmetric (odd-term) contributions are weak.

Cheating?

$$H_p = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p$$

$$H_{2X} = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^2$$

Accidental overlap

$$\left|\langle g_p | g_{2X}\rangle\right| \propto N^{-1/2} >> e^{-aN}$$

Significant overlap, thus embedding the answer

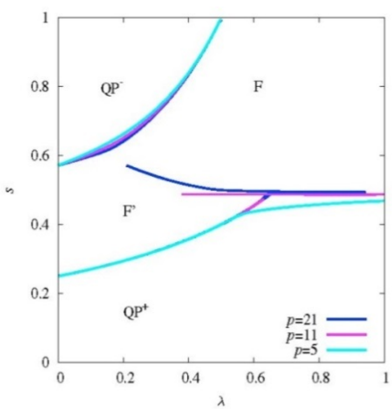
Bapst, Semerjian

k-body *XX* interactions

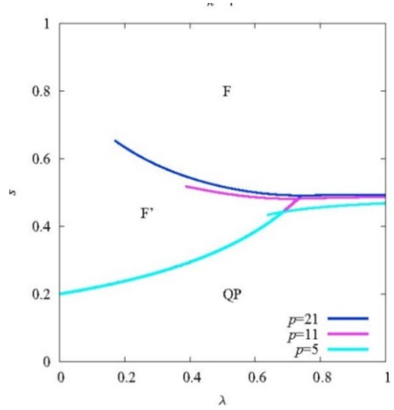
$$H(s,\lambda) = -s\left[\lambda N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - (1-\lambda)N\left(\frac{1}{N}\sum_{i}\sigma_i^x\right)^k\right] - (1-s)\sum_{i=1}^{N}\sigma_i^x$$

Overlap between ground states

$$\left|\langle g_p | g_{kX}\rangle\right| \propto \begin{cases} N^{-1/2} & (k \text{ even}) \\ e^{-aN} & (k \text{ odd}) \end{cases}$$



$k=3$



$k=4$

More complex problem: Random interactions

Hopfield model: non-trivial ground state

$$H_{\text{Hopfield}} = -\sum_{i_1<i_2<\ldots<i_k} J_{i_1 i_2 \ldots i_k} \sigma_{i_1}^z \sigma_{i_2}^z \cdots \sigma_{i_k}^z \quad (k = 2, 3, 4, \ldots)$$

$$J_{i_1 i_2 \ldots i_k} = N^{-k+1} \sum_{\mu=1}^{p} \xi_{i_1}^{\mu} \xi_{i_2}^{\mu} \cdots \xi_{i_k}^{\mu} \qquad (\xi = \pm 1)$$

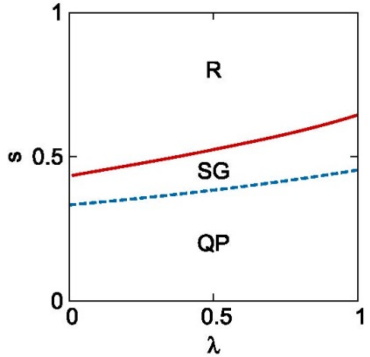
Quantum annealing with *XX* interactions

$$H(s) = s\left[\lambda H_{\text{Hopfield}} + (1-\lambda) N \left(\frac{1}{N}\sum_i \sigma_i^x\right)^2\right] - (1-s)\sum_{i=1}^{N} \sigma_i^x$$

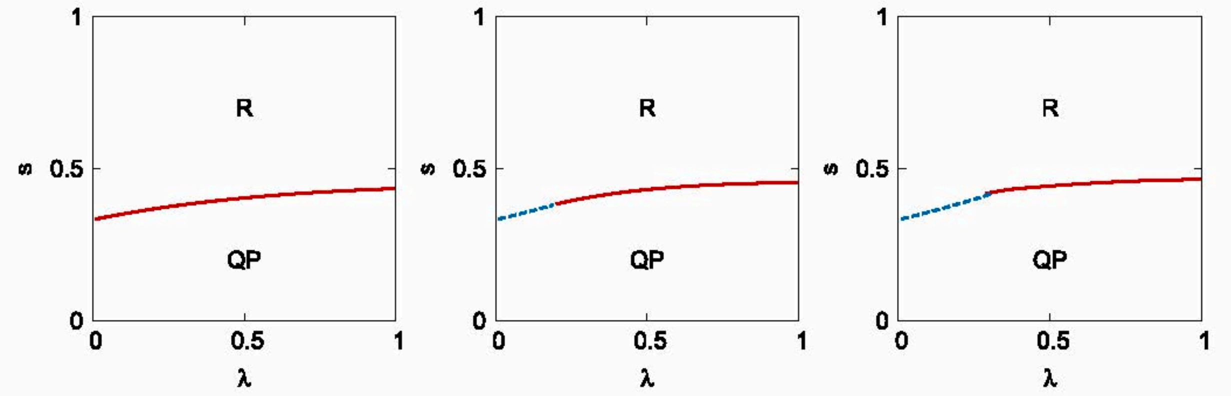
Finite p Same as the non-random case: can avoid 1st order.

Extensive p

$$k = 2, \quad p = 0.04N$$



$$-\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z$$



$$-\sum_{i_1<i_2<\ldots<i_k} J_{i_1 i_2 \ldots i_k}\sigma_{i_1}^z\sigma_{i_2}^z\cdots\sigma_{i_k}^z$$

$$p = 0.04N^{k-1} \quad k = 3, 4, 5$$

Why does the AF *XX* term work?

$$H_{\text{problem}} - (1-s)\sum_i \sigma_i^x + cN\left(\frac{1}{N}\sum_i \sigma_i^x\right)^2$$

$$\begin{pmatrix} * & - & - & - \\ - & * & 0 & - \\ - & 0 & * & 0 \\ - & - & 0 & * \end{pmatrix}$$

- c=0: Stoquastic (fixed sign in off-diagonal)

  Can be mapped to classical Ising and simulated efficiently.

$$\begin{pmatrix} * & + & - & + \\ + & * & 0 & - \\ - & 0 & * & - \\ + & - & - & * \end{pmatrix}$$

- c>0: Non-stoquastic (both signs in off-diagonal)

  Difficult to efficiently simulate classically.

Quantum effects: strong

Conclusion

- 1st order transition → 2nd order transition

  by the antiferromagnetic *XX* interactions.

  Exponential reduction in computation time.

- These are the first examples where an intrinsic quantum speedup has been found out of the antiferromagnetic XX interactions.

- "Intrinsic quantum speedup" means an exponential reduction of computation time by non-stoquastic (classically unable to simulate) terms.