Quantum annealing by ferromagnetic interaction with the mean-field scheme

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Abstract

Quantum annealing is a novel method for combinatorial optimization problems. In this paper, we discuss the appropriate choice of quantum fluctuations in quantum annealing. The existence of room of choices of quantum fluctuations is an advantage of quantum annealing over simulated annealing. We consider the ferromagnetic interaction as a source of quantum fluctuations. Using the mean-field annealing scheme, we show that quantum annealing by ferromagnetic interaction is more efficient than the conventional quantum annealing and simulated annealing in the ground state search of the random-field Ising model.

1 Introduction

Close relations between statistical physics, information theory and computer science have aroused our attention to the importance of the ground state search of random Ising models [1]. The ground state search of random Ising models generally needs the time which grows exponentially with the number of Ising variables. Hence it is, in general, intractable to find out the ground state in Ising systems even with hundreds of spins. A celebrated prescription to treat such difficult problems approximately is simulated annealing [2]. One introduces a finite temperature which follows the Maxwell-Boltzmann distribution. If the system is cooled slowly, the equilibrium is followed closely and the ground state is reached at the zero temperature limit. Quantum annealing for Ising systems was proposed by analogy with simulated annealing [3]. One introduces quantum fluctuations, which is initially so strong as to overwhelm the Ising system. The ground state of this initial system is supposed to be given. Lowering the magnitude of quantum fluctuations slowly with time, the

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quantum spin state stays near the ground state at each time and one can finally obtain the ground state of the original Ising system when the additional quantum fluctuations vanish.

Most studies on quantum annealing so far have used the transverse Ising model, and compared quantum annealing with simulated annealing [4,5]. However quantum annealing has a useful feature that there are possible choices of the source of quantum fluctuations, which does not exist in simulated annealing. In the present paper, we focus on the ferromagnetic interaction as the source of quantum fluctuations and compare conventional quantum annealing, quantum annealing by ferromagnetic interaction, and simulated annealing.

The model employed in this paper is the random-field Ising model. The goal of the optimization problem is to find out the ground state of this model. It has been reported that conventional quantum annealing works much better than simulated annealing for paramagnetic ground states of the random-field Ising model, but this is not the case for ferromagnetic ground states [6]. In this paper, we implement quantum annealing and simulated annealing by means of the mean-field annealing scheme [7], and show that residual error after quantum annealing by ferromagnetic interaction is definitely lower than conventional quantum annealing and simulated annealing even for ferromagnetic ground states. Consequently, we conclude that the reduced performance reported in ref. [6] does not come from an intrinsic feature in quantum annealing.

In the following part of this paper, we explain quantum annealing by ferromagnetic interaction as well as conventional quantum annealing in Sec. 2. Section 3 is devoted to explanation of the mean-field annealing method. Results of the mean-field annealing are presented in Sec. 4. This paper is concluded in Sec. 5.

2 Quantum annealing by ferromagnetic interaction

The random-field Ising model is written as

$$\mathcal{H}_{\rm pot}^{\rm RFIM} = -\sum_i h_i \sigma_i^z - J \sum_{<ij>} \sigma_i^z \sigma_j^z, \tag{1}$$

where σ_i^z is the z-component of the Pauli matrix, and either $+1$ or -1 is assigned to each random field h_i with the same probability at each site. We assume the square lattice system in two dimension throughout this paper. Regarding the Ising Hamiltonian as the potential term, we introduce a kinetic term which does not commute with the potential term. In conventional

quantum annealing, the kinetic term is given by the transverse field,

$$\mathcal{H}_{\rm kin}^{\rm TF} = -\sum_i \sigma_i^x, \tag{2}$$

where σ_i^x is the x-component of the Pauli matrix. In the present paper, we instead introduce the following kinetic term,

$$\mathcal{H}_{\rm kin}^{\rm FI} = -\sum_i \sigma_i^x - \sum_{<ij>} \sigma_i^x \sigma_j^x. \tag{3}$$

We note that the term of the transverse field in this equation is necessary in order to make the ground state of the kinetic Hamiltonian unique. We assume that the strengths of the transverse field and the interaction are the same for simplicity.

We then construct a time-dependent Hamiltonian as [8]

$$\mathcal{H}(t) = \left(1 - \frac{t}{\tau}\right) \mathcal{H}_{\rm kin}^{\rm TF,FI} + \frac{t}{\tau} \mathcal{H}_{\rm pot}^{\rm RFIM}, \tag{4}$$

where τ is a given parameter to determine the time consumed for annealing. This time-dependent Hamiltonian is identical with the kinetic Hamiltonian at $t = 0$, whereas it is nothing but the potential Hamiltonian at $t = \tau$. We take the initial state to be the ground state of the kinetic Hamiltonian. The ground states of $\mathcal{H}_{\rm kin}^{\rm TF}$ and $\mathcal{H}_{\rm kin}^{\rm FI}$ are the same and given by the state in which all spins are aligned along the x-axis.

The time evolution of the state in quantum mechanics is governed by the Schrödinger equation. Then the quantum adiabatic theorem guarantees that the final state converges to the ground state of the final Hamiltonian $\mathcal{H}(\tau)$ in the limit of infinitely slow time evolution. In order to implement quantum annealing, one has to follow time evolution of the quantum state. However it is a difficult task to solve the Schrödinger equation for large systems. Hence we employ another rule of dynamics, which we describe in the next section.

3 Mean-field annealing

In the Weiss mean-field approximation, the density operator of the system is decomposed into the direct product of the density operators of each site, $\rho \cong \prod_i \rho_i$. Let us focus our attention on the spin state at site i, provided that local magnetizations at sites j connected to i, $m_j^z = {\rm Tr}_j(\sigma_j^z \rho_j)$ and $m_j^x = {\rm Tr}_j(\sigma_j^x \rho_j)$, are given. Remark that ${\rm Tr}_j$ indicates the trace over the states at site

j. The variational free energy in the mean-field approximation is defined by $\mathcal{F}_{\rm trial} = {\rm Tr}(\mathcal{H}(t)\rho) + T{\rm Tr}(\rho \ln \rho)$, where T indicates the temperature. Applying the variational principle, one can obtain the optimized density operator of site i as $\rho_i^{\rm opt} = e^{-\beta\mathcal{H}_i}/{\rm Tr}_i(e^{-\beta\mathcal{H}_i})$ with the inverse temperature β, where

$$\mathcal{H}_i = \left(1 - \frac{t}{\tau}\right)\left(-\sigma_i^x - \sum_{j\in\mathcal{S}(i)} \sigma_i^x m_j^x\right) + \frac{t}{\tau}\left(-h_i\sigma_i^z - J\sum_{j\in\mathcal{S}(i)} \sigma_i^z m_j^z\right). \quad (5)$$

Here $\mathcal{S}(i)$ denotes the set of sites connected to i. In the case of zero temperature, the above density operator is simply written in terms of $|{\rm g}\rangle_i$, the normalized ground state of $\mathcal{H}_i$, as $\rho_i^{\rm opt} = |{\rm g}\rangle_{ii}\langle{\rm g}|$.

Using the optimized density operator, one can obtain the local magnetizations at site i, $m_i^z = {\rm Tr}_i(\sigma_i^z\rho_i^{\rm opt})$ and $m_i^x = {\rm Tr}_i(\sigma_i^x\rho_i^{\rm opt})$. We successively shift the site under attention after the computation of local magnetizations. The computation in the static mean-field approximation is repeated until local magnetizations become stable.

In order to implement quantum annealing in the mean-field approximation, one needs to change the time t dynamically. In our study, we change t by one from $t = 0$ to $t = \tau$ whenever the site under attention is shifted over the whole system. To implement simulated annealing, we suppose $t = 0$ in Eq. (5) and change the temperature from an appropriate value to zero using τ steps.

4 Results

We performed quantum annealing by transverse field, quantum annealing by ferromagnetic interaction, and simulated annealing by means of the mean-field annealing scheme for the random-field Ising model. For each instance of the random-field configuration, we evaluated residual energies of the states after mean-field annealing by the formula, $E_{\rm res}(\tau) = E(\tau) - E_{\rm gs}$, where $E(\tau)$ is the energy of the state after mean-field annealing with given τ and $E_{\rm gs}$ is the exact ground energy obtained by an algorithm described in ref. [9].

We show residual energies after quantum annealing by transverse field, quantum annealing by ferromagnetic interaction, and simulated annealing in Fig. 1. Figures (a) and (b) correspond to the result with $J = 0.6$ and $J = 2.0$, respectively, of the coupling constant in the random-field Ising model. In both cases, residual energy of quantum annealing by transverse field is the highest for large τ. The mean-field approximation seems to affect conventional quantum annealing more seriously than simulated annealing. However the effect of ferromagnetic interaction outweighs the disadvantage of conventional quan-

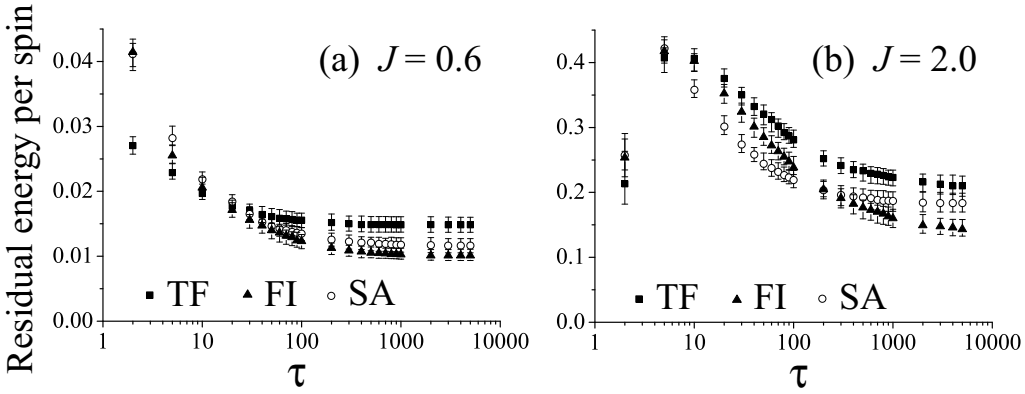

Fig. 1. Residual energies of the random-field Ising model with the coupling constant (a) $J = 0.6$ and (b) $J = 2.0$. Quantum annealing by transverse field (TF), quantum annealing by ferromagnetic interaction (FI), and simulated annealing (SA) are implemented by means of the mean-field annealing scheme. The square lattice system with the size 100×100 in two-dimension is investigated. 80 samples of random-field are taken into account. Symbols in the figures indicate median values, and error bars are for the upper and lower half medians.

tum annealing. It is evident that residual energy of quantum annealing by ferromagnetic interaction is the lowest for large τ in both figures.

Comparing figures 1(a) and 1(b), we notice that residual energy of quantum annealing by ferromagnetic interaction is lowered more remarkably in Fig. 1(b). The random-field Ising model with the present system size has the paramagnetic ground state when $J = 0.6$ and the ferromagnetic one when $J = 2.0$. An earlier Monte-Carlo studies have shown that the decay rate of residual energy after conventional quantum annealing is as slow as simulated annealing for ferromagnetic ground states though it is faster than simulated annealing for paramagnetic ground states. Our results imply that quantum annealing by ferromagnetic interaction works better for ferromagnetic ground states. In fact, we have also confirmed by the Bethe-type mean-field annealing that quantum annealing by ferromagnetic interaction is extremely good for the search for ferromagnetic ground states [10]. Therefore the reduced efficiency of conventional quantum annealing for ferromagnetic ground states reported in ref. [6] may be an artifact of the specific implementation of quantum annealing. An appropriate choice of quantum fluctuations would accelerate the decay rate of residual energy.

5 Conclusion

The existence of flexible choices of quantum fluctuations is an outstanding feature of quantum annealing. In this paper, we have focused on the transverse-

ferromagnetic interaction as a source of quantum fluctuations. In order to reveal the validity of quantum annealing by ferromagnetic interaction, we implemented quantum annealing for the random-field Ising model by means of the mean-field annealing scheme. We found that quantum annealing by ferromagnetic interaction clearly reduces residual energy in comparison with conventional quantum annealing by transverse field and simulated annealing. In addition, results of the mean-field annealing show an indication that quantum annealing by ferromagnetic interaction is more effective for ferromagnetic ground states than for paramagnetic ones. The present results suggest that previously observed inefficiency [6,11] of quantum annealing can be circumvented by choosing appropriate quantum fluctuations.

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