Multicritical point of spin glasses *

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We present a theoretical framework to accurately calculate the location of the multicritical point in the phase diagram of spin glasses. The result shows excellent agreement with numerical estimates. The basic idea is a combination of the duality relation, the replica method, and the gauge symmetry. An additional element of the renormalization group, in particular in the context of hierarchical lattices, leads to impressive improvements of the predictions.

1. INTRODUCTION

Identification of the precise location of the multicritical point is an important theoretical challenge in the physics of spin glasses not only because of its mathematical interest but also for the practical purpose of reliable analyses of numerical data. The method of duality is a standard tool to derive th exact location of a critical point in pure ferromagnetic systems in two dimensions. However, the existence of randomness in spin glasses hampers a direct application of the duality.

We have nevertheless developed a theory to achieve the goal by using the combination of the replica method, the duality applied to the replicated system, the gauge symmetry, and the renormalization group [1–6]. The result shows excellent agreement with numerical estimates. The analysis on hierarchical lattices plays a crucial role in the development of the theory, in particular in the introduction of the renormalization group, by which systematic improvements can be achieved.

2. MULTICRITICAL POINT

Let us consider the $\pm J$ Ising model defined by the Hamiltonian,

$$H = -\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j, \tag{1}$$

where σ_i is the Ising spin and J_{ij} denotes the quenched random coupling. The sign of J_{ij}, i.e. $J_{ij}/J = \tau_{ij}$, follows the distribution

$$\begin{aligned} P(\tau_{ij}) &= p\delta(1-\tau_{ij}) + (1-p)\delta(1+\tau_{ij}) \\ &= \frac{\exp(K_p\tau_{ij})}{2\cosh K_p}\left\{\delta(1-\tau_{ij}) + \delta(1+\tau_{ij})\right\}, \end{aligned} \tag{2}$$

*Dedicated to Prof. A. Nihat Berker on the occasion of his sixtieth birthday.

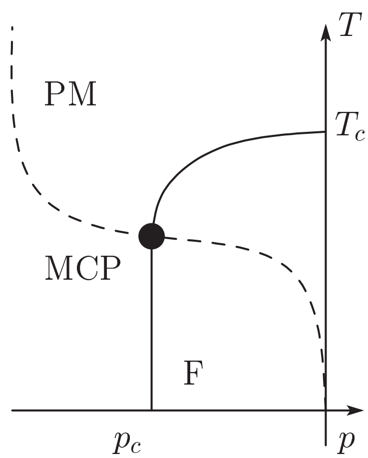

Figure 1. A typical phase diagram of the $\pm J$ Ising model in two dimensions. The multicritical point (MCP) is described by a black dot and the Nishimori line is drawn dashed.

where $\exp(-2K_p) = (1-p)/p$. The multicritical point is believed to lie on the Nishimori line (NL) defined by $K_p = \beta J$, where β is the inverse temperature. See Fig. 1. The restriction to the NL simplifies the problem due to the gauge symmetry [7,8].

According to the initial theory that uses the replica method, duality and gauge symmetry [1–4], the value of p_c for the multicritical point satisfies

$$H(p_c) = \frac{1}{2}, \tag{3}$$

where $H(p)$ is the binary entropy, $-p\log_2 p - (1-p)\log_2(1-p)$, for self-dual lattices. Equation (3) is solved to give $p_c = 0.8900$, which is in reasonable agreement with numerical estimates. The theory has also been extended to a pair of mutually dual lattices with p_{c1} and p_{c2} for respective multicritical points. The result is

$$H(p_{c1}) + H(p_{c2}) = 1. \tag{4}$$

Hinczewski and Berker, however, found $H(p_1) + H(p_2) = 1.0172, 0.9829, 0.9911$ for three pairs of mutually dual hierarchical lattices [9]. Their values are correct to the decimal points shown above as one can carry out numerically exact renormalization group calculations on hierarchical lattices. Thus Eq. (4) is a good approximation but not quite exact, at least for hierarchical lattices.

3. REPLICA AND DUALITY

Let us give a very brief summary of the theory that leads to Eqs. (3) and (4). We generalize the usual duality argument to the n-replicated $\pm J$ Ising model.

We define the edge Boltzmann factor x_k ($k = 0, 1, \cdots, n$), which represents the configuration-averaged Boltzmann factor for interacting spins with k antiparallel spin pairs among n

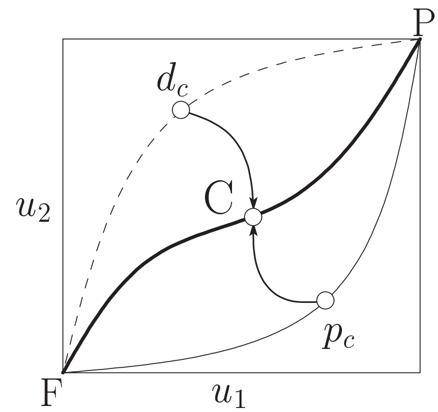

Figure 2. A schematic picture of the renormalization flow and the duality for the replicated $\pm J$ Ising model.

nearest-neighbour pairs for a bond (edge). The duality gives the following relationship between the partition functions on the original and dual lattices with different values of the edge Boltzmann factors

$$Z_n(x_0, x_1, \cdots, x_n) = Z_n(x_0^*, x_1^*, \cdots, x_n^*), \tag{5}$$

where we have assumed self duality of the lattice in that both sides share the same function Z_n. The dual edge Boltzmann factors x_k^* are defined by the discrete multiple Fourier transforms of the original edge Boltzmann factors, which are simple combinations of plus and minus of the original Boltzmann factors in the case of Ising spins.

It turns out useful to focus our attention to the principal Boltzmann factors x_0 and x_0^*, which are the most important elements of the theory. Their explicit forms are

$$x_0(K, K_p) = \frac{\cosh(nK + K_p)}{\cosh K_p}, \quad x_0^*(K, K_p) = \left(\sqrt{2}\cosh K\right)^n, \tag{6}$$

where $K = \beta J$. We extract these principal Boltzmann factors from the partition functions in Eq. (5), which amounts to measuring the energy from the all-parallel spin configuration. Then, using the normalized edge Boltzmann factors $u_j = x_j/x_0$ and $u_j^* = x_j^*/x_0^*$, we have

$$x_0(K, K_p)^{N_B} z_n(u_1, u_2, \cdots, u_n) = x_0^*(K, K_p)^{N_B} z_n(u_1^*, u_2^*, \cdots, u_n^*), \tag{7}$$

where $z_n(u_1, \cdots)$ and $z_n(u_1^*, \cdots)$ are defined as $Z_n/x_0^{N_B}$ and $Z_n/(x_0^*)^{N_B}$ and N_B is the number of bonds.

We now restrict ourselves to the NL, $K = K_p$. Figure 2 shows the relationship between the curves $(u_1(K), u_2(K), \cdots, u_n(K))$ (the thin curve) and $(u_1^*(K), u_2^*(K), \cdots, u_n^*(K))$ (the dashed curve). The arrows emanating from both curves represent the renormalization flows toward the fixed point C.

The ordinary duality argument identifies the critical point under the assumption of a unique phase transition. We can obtain the critical point as the fixed point of the duality transformation using the fact that the partition function is a single-variable function. In other words, the thin curve would overlap with the dashed line for such a case.

In the present random case, on the other hand, since z_n is a multivariable function, there is no fixed point of the duality in the strict sense which satisfies n conditions simultaneously, $u_1(K) = u_1^*(K), u_2(K) = u_2^*(K), \cdots, u_n(K) = u_n^*(K)$. This is in sharp contrast to the non-random Ising model. We nevertheless assume that $x_0(K,K) = x_0^*(K,K)$ may give the precise location of the multicritical point because, when the number of variables of z_n in Eq. (7) is unity ($n = 1$), the fixed point condition $u_1 = u_1^*$ implies $x_0 = x_0^*$. This relation, in the limit of $n \to 0$ in the spirit of the replica method, leads to Eq. (3). A straightforward generalization to mutual dual cases gives Eq. (4).

4. RENORMALIZATION GROUP ON HIERARCHICAL LATTICES

The renormalization group provides us with an additional point of view, especially on hierarchical lattices. Let us remember the following features of the renormalization group: (i) The critical point is attracted toward the unstable fixed point. (ii) The partition function does not change its functional form by the renormalization on hierarchical lattices; only the values of arguments change. Therefore the renormalized system also has a representative point in the same space $(u_1(K), u_2(K), \cdots, u_n(K))$ as in Fig. 2. The renormalization flow from the critical point p_c reaches the fixed point C, $(u_1^{(\infty)}, u_2^{(\infty)}, \cdots, u_n^{(\infty)})$. Here the superscript means the number of renormalization steps. There is a point d_c related to p_c by the duality, which is expect to also reach the same fixed point C since p_c and d_c represent the same critical point due to Eq. (5). Considering the above property of the renormalization flow as well as the duality, we find that the duality relates two trajectories of the renormalization flow from p_c and from d_c. The same applies to the whole part of both curves, thin and dashed. In other words, after a sufficient number of renormalization steps, the thin curve representing the original system and the dashed curve for the dual system both approach the common renormalized system depicted as the bold curve in Fig. 2, which goes through the fixed point C.

The partition function is then expected to become a single-variable function along the bold curve. This fact enables us to improve the method so that the exact location of the multicritical point is obtained asymptotically, which can be given by $x_0^{(s\to\infty)}(K) = x_0^{*(s\to\infty)}(K)$. If we regard $x_0(K) = x_0^*(K)$ as the zeroth approximation for the location of the multicritical point, it is expected that $x_0^{(1)}(K) = x_0^{*(1)}(K)$ is the first approximation and can lead to more precise results than $x_0(K) = x_0^*(K)$ does.

Our method by the duality analysis in conjunction with the renormalization group indeed has given the results in excellent agreement with the exact estimations within numerical errors on several self-dual hierarchical lattices as summarized in Table 1.

5. FURTHER DEVELOPMENTS

The above method has also been generalized to be applicable to Bravais lattices [6]. Let us take an example of the square lattice. Instead of the iterative renormalization, we consider to sum over a part of the spins, to be called a cluster, on the square lattice as shown in Fig. 3 to incorporate many-body effects such as frustration inherent in spin

p_c (without RG)	p_c (with RG)	p_c (numerical)
0.8900	0.8920	0.8915(6)
0.8900	0.8903	0.8903(2)
0.8900	0.8892	0.8892(6)
0.8900	0.8895	0.8895(6)
0.8900	0.8891	0.8890(6)

Table 1
Comparison of the methods with and without RG and numerical estimations for several self-dual hierarchical lattices [5].

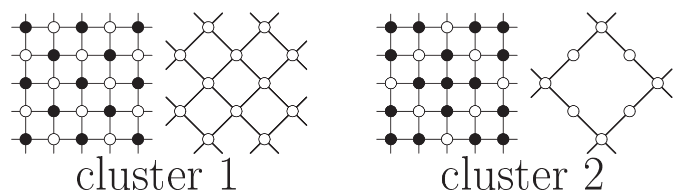

Figure 3. The basic clusters used on the square lattice. The spins marked black on the original lattice are traced out instead of the iterative renormalization.

glasses. To this end, we define the principal Boltzmann factors $x_0^{(s)}$ and its dual $x_0^{*(s)}$ as those with all spins surrounding the cluster in the up state. We assume that a single equation gives the accurate location of the multicritical point $x_0^{(s)}(K) = x_0^{*(s)}(K)$, where the superscript s stands for the type of the cluster. Recent numerical investigations on the square lattice have given $p_c = 0.89081(7)$ [10], $p_c = 0.89083(3)$ [11] and $p_c = 0.89061(6)$ [12], while the present method has estimated $p_c = 0.890725$ by cluster 1 of Fig. 3, and $p_c = 0.890822$ by cluster 2 [6]. If we deal with clusters of larger sizes, the new method is expected to show systematic improvements toward the exact answer from the point of view of renormalization.

The method of the renormalization group is applicable also away from the NL. For example, the slope of the phase boundary at the pure ferromagnetic limit has been estimated to be $1/T_c \times dT/dp \approx 3.2091\cdots$ on the square lattice by perturbation [14]. This result is applicable also to any self-dual hierarchical lattices. The present method with the renormalization group taken into account shows that this is not the case. The result depends on the type of lattice, e.g. $3.2786\cdots$ and $3.4390\cdots$ [15].

6. CONCLUSION

The hierarchical lattices provide a very effective platform to test new ideas as has been exemplified in the present study. Investigations are notoriously hard for spin glasses on finite-dimensional systems both analytically and numerically. On hierarchical lattices, on the other hand, numerically exact calculations can be carried out, and, in addition, hierarchical lattices share many features with finite-dimensional systems in contrast to mean-field systems. Analytical methods can also be implemented with relative ease on hierarchical lattices, which leads to the significant improvements in the prediction of the location of the multicritical point. Hierarchical lattices will continue to play key roles in the studies of spin glass and other complex systems.

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