Comment on "Statistical mechanics of CDMA multiuser demodulation"

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In a recent paper [1], Tanaka formulated and solved a model of CDMA multiuser demodulation by applying the theory of spin glasses, the replica method in particular. It is shown in the present comment that some of his results can be derived without recourse to the replica method; conclusions from the somewhat tricky replica method are justified rigorously. It also becomes clear that there is no finite-size effect in the internal energy for the MPM demodulator and that the structure of the phase space is simple; the latter goes beyond the local stability argument of the AT line analysis.

The first of our result is a generalization of the identity $m = q$ for the MPM demodulator derived by Tanaka using the replica method: The functional identity $P_m(x) = P_q(x)$ will be proved for MPM. Here $P_m(x)$ $(-1 \le x \le 1)$ is the distribution function of the order parameter corresponding to magnetization and $P_q(x)$ denotes the distribution of the spin glass order parameter.

The proof follows the same line as in the spin glass case [2, 3]. The function $P_m(x)$ is defined as

$$P_m(x) = \frac{1}{2^{Np+N}}\left(\frac{\beta_s}{2\pi}\right)^{p/2}\sum_\eta\int\prod_t dr^t\sum_\xi\exp\left\{-\frac{\beta_s}{2}\sum_t(r^t-\frac{1}{\sqrt{N}}\sum_i\eta_i^t\xi_i)^2\right\}$$
$$\cdot\ \frac{\sum_s\delta(x-\frac{1}{N}\sum_i\xi_iS_i)\exp\left\{-\frac{\beta}{2}\sum_t(r^t-\frac{1}{\sqrt{N}}\sum_i\eta_i^tS_i)^2\right\}}{\sum_s\exp\left\{-\frac{\beta}{2}\sum_t(r^t-\frac{1}{\sqrt{N}}\sum_i\eta_i^tS_i)^2\right\}}. \quad (1)$$

Here r^t is the received signal scaled by $\sqrt{N}$, $r^t = y^t/\sqrt{N}$, η_i^t is the spreading code sequence, ξ_i is the original information symbol of user i, and S_i is the dynamical variable for demodulation. Note that the order parameter m in Tanaka's article stands for the average overlap of the original symbol ξ_i and the thermal expectation value of the demodulation variable S_i. Then the function $P_m(x)$ represents the distribution of his order parameter. The function $P_m(x)$ is equivalent to the usual magnetization appearing in the theory of spin glasses: One can confirm it by the gauge transformation $\eta_i^t \to \eta_i^t\xi_i$ and $S_i \to S_i\xi_i$ for given ξ_i. Then the expression after $\sum_\xi$ in eq. (1) becomes independent of ξ_i, so that this sum over ξ can be eliminated. The result is that $P_m(x)$ measures the magnetization distribution of the system with dynamical variables S_i.

The distribution function $P_q(x)$ is defined similarly with the second line of eq. (1) replaced by

$$\frac{\sum_{s,\sigma} \delta(x - \frac{1}{N}\sum_i S_i\sigma_i) \exp\left\{-\frac{\beta}{2}\sum_t (r^t - \frac{1}{\sqrt{N}}\sum_i \eta_i^t S_i)^2 - \frac{\beta}{2}\sum_t (r^t - \frac{1}{\sqrt{N}}\sum_i \eta_i^t \sigma_i)^2\right\}}{\sum_{s,\sigma} \exp\left\{-\frac{\beta}{2}\sum_t (r^t - \frac{1}{\sqrt{N}}\sum_i \eta_i^t S_i)^2 - \frac{\beta}{2}\sum_t (r^t - \frac{1}{\sqrt{N}}\sum_i \eta_i^t \sigma_i)^2\right\}}. \quad (2)$$

Comparison of eqs. (1) and (2) immediately reveals the relation $P_m(x) = P_q(x)$ under the MPM condition $\beta = \beta_{\rm s}$ since one of the sums in the denominator of eq. (2) cancels with the sum over ξ in the first line of eq. (1).

It is well established that the distribution of magnetization is simple, that is, $P_m(x)$ is composed of at most two delta functions, and therefore the function $P_q(x)$ should also be simple. This implies that the structure of the phase space is simple and there is no replica symmetry breaking for MPM demodulator, in agreement with the AT stability analysis in replica calculations. Indeed the simple structure of $P_q(x)$ means more: The AT analysis is for local stability against RSB perturbations whereas the simple $P_q(x)$ suggests that the whole phase space is simple, a result on the global structure. The identity $m = q$ derived by Tanaka can be proved by integration of $xP_m(x)$ and $xP_q(x)$ over $0 \le x \le 1$.

Along the same line of reasoning, it is possible to evaluate the internal energy U for MPM demodulator without using replicas. The definition of the internal energy is almost the same as eq. (1), the only difference being that the delta function in the numerator of the second line is replaced with partial derivative $\partial/\partial\beta$. It then follows that the sums over ξ and S cancel under the MPM condition $\beta = \beta_{\rm s}$. It is straightforward to evaluate the resulting expression by carrying out the integral over r^t first and then taking derivative by β. The result is

$$U = -\frac{p}{2\beta}. \quad (3)$$

This agrees with the replica calculation as can be verified by taking the β-derivative of the free energy, Tanaka's eq. (10). It is remarkable that our eq. (3) is applicable to any finite-size system whereas Tanaka's energy is for the thermodynamic limit. Agreement of the two results means that finite-size effects are completely absent in the MPM internal energy averaged over quenched randomness.

We have shown that some of the important results of Tanaka can be derived without replicas as long as the MPM demodulator is concerned. Application of the present method to the MAP case is an interesting but difficult future problem.

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REFERENCES

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