Exact location of the multicritical point for finite-dimensional spin glasses: A conjecture

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Abstract. We present a conjecture on the exact location of the multicritical point in the phase diagram of spin glass models in finite dimensions. By generalizing our previous work, we combine duality and gauge symmetry for replicated random systems to derive formulas which make it possible to understand all the relevant available numerical results in a unified way. The method applies to non-self-dual lattices as well as to self dual cases, in the former case of which we derive a relation for a pair of values of multicritical points for mutually dual lattices. The examples include the $\pm J$ and Gaussian Ising spin glasses on the square, hexagonal and triangular lattices, the Potts and Z_q models with chiral randomness on these lattices, and the three-dimensional $\pm J$ Ising spin glass and the random plaquette gauge model.

1. Introduction

Properties of finite-dimensional spin glasses are still under debate although the problem is essentially settled for the mean-field model [1]. Outstanding problems for finite-dimensional spin glasses include the existence or absence of spin glass phase and whether or not the mean-field picture of the spin glass phase applies. Another interesting, but less extensively studied, issue is the structure of the phase diagram, in particular where precisely the multicritical point is located and what the values are for the critical exponents characterizing the system behaviour at and away the multicritical point. The present paper discusses this problem of the location of the multicritical point for finite-dimensional spin glass models by analytical methods.

A number of numerical investigations on this problem exist for various lattices. However, it has been quite difficult to derive analytical results for regular finite-dimensional lattices until a few years ago when we succeeded in devising a method to predict the exact locations of the multicritical points for the square lattice Ising and Potts models and four-dimensional random plaquette gauge model using duality, gauge symmetry and the replica method [2, 3, 4]. In the present paper we generalize this theoretical framework so that it is applicable to a pair of mutually-dual lattices, for which our theory relates the pair of values of multicritical points of the two lattices.

The logic of our theory includes a step which is yet to be justified rigorously, and hence the status of our result is a conjecture at this moment. Nevertheless, our theory

enables us to understand all the relevant available numerical data for the multicritical points derived independently by a number of authors. Also our theory satisfies necessary conditions which the exact solution should satisfy.

This paper is organized as follows. In the next section, we recall the basic formulation which was developed in our previous studies in order to fix the notation and set the stage for further developments in the following sections. In section 3, a generalization of the theory to non-self-dual Z_2 (Ising) models with randomness is discussed. This argument is followed by section 4, in which we further generalize the arguments to Z_q systems with chiral randomness. The final section is devoted to conclusion and discussions.

2. Self-dual Z_2 models

In this section we briefly review the duality arguments in [3, 4] applied to the two-dimensional ($2d$) $\pm J$ random bond Ising model on the square lattice and the $4d$ random Z_2 lattice gauge model, by which we fix the notation and set a stage for generalizations in the following sections. After recalling the duality arguments for non-random systems, we apply the idea to random systems.

2.1. Duality of non-random models

We first elucidate the duality of a non-random Z_2 (Ising) system [5]. Let us prepare a d-dimensional lattice and assign Z_2 spins on $r-1$ dimensional elements $\boldsymbol{x}$ on the lattice, which we denote by $S_{\boldsymbol{x}}$. We consider a model on the lattice whose Hamiltonian is given by

$$H = -J \sum_{C} \prod_{\boldsymbol{x} \in \partial C} S_{\boldsymbol{x}}, \tag{1}$$

where C is the r-dimensional element on the lattice and ∂C is its boundary of dimension $r-1$ [5]. Let $u_{\pm 1}$ denote the Boltzmann factor for an element C,

$$u_{\pm 1}(K) \equiv e^{\pm K}, \qquad \text{for} \prod_{\boldsymbol{x} \in \partial C} S_{\boldsymbol{x}} = \pm 1, \tag{2}$$

where $K \equiv \beta J$. For the case of the usual Ising model ($r = 1$), $u_1(K)$ is the bond (edge) Boltzmann factor for parallel spins at both ends and $u_{-1}(K)$ is for anti-parallel spins. Then the partition function $\mathcal{Z}$ is a function of $u_{\pm 1}$; $\mathcal{Z} = \mathcal{Z}\{u_1(K), u_{-1}(K)\}$.

The dual model is defined on the dual lattice (the definition of which is given in [5]). The dual Boltzmann factor for the dual element C^* is defined by the two-component Fourier transformation of $u_{\pm 1}$ [6],

$$u^*_{\pm 1}(K) \equiv \frac{u_1(K) \pm u_{-1}(K)}{\sqrt{2}} = \frac{e^K \pm e^{-K}}{\sqrt{2}}. \tag{3}$$

For the present Hamiltonian (1), the dual Hamiltonian is given by

$$H^* = -J \sum_{C^*} \prod_{\boldsymbol{x} \in \partial C^*} S_{\boldsymbol{x}}, \tag{4}$$

which is of the same form as equation (1). C^* is the dual element of C and has dimension $d-r$. Next we derive the dual expression of the partition function for general dual pairs. The partition function as a function of u has the following property,

$$\mathcal{Z}_{\rm orig}\{u_1(K), u_{-1}(K)\} = 2^a \mathcal{Z}_{\rm dual}\{u_1^*(K), u_{-1}^*(K)\}, \tag{5}$$

Here $\mathcal{Z}_{\rm orig}$ is the partition function of the original model and $\mathcal{Z}_{\rm dual}$ is for the dual model. The symbol a is a constant determined by the numbers of the elements on the lattice. (See appendix.)

We give several examples of dual pairs below, which would be helpful to understand the definition of the dual transformation. The arrows indicate duality relations.

- $(d = 2, r = 1)$ $2d$ Ising model on the square lattice $\leftrightarrow$ $2d$ Ising model on the square lattice (self-dual)
- $(d = 2, r = 1)$ $2d$ Ising model on the triangular lattice $\leftrightarrow$ $2d$ Ising model on the hexagonal lattice
- $(d = 4, r = 2)$ $4d$ Z_2 lattice gauge model on the hypercubic lattice $\leftrightarrow$ $4d$ Z_2 lattice gauge model on the hypercubic lattice (self-dual)
- $(d = 3, r = 1)$ $3d$ Ising model on the cubic lattice $\leftrightarrow$ $3d$ Z_2 lattice gauge model on the cubic lattice

So far the discussions have not been restricted to self-dual models.

For self-dual cases, $\mathcal{Z}_{\rm orig}$ and $\mathcal{Z}_{\rm dual}$ are the same function and the prefactor 2^a on the right hand side in (5) becomes a trivial constant which is negligible in the thermodynamic limit (and is omitted in the following). Hence (5) is simplified to

$$\mathcal{Z}\{u_1(K), u_{-1}(K)\} = \mathcal{Z}\{u_1^*(K), u_{-1}^*(K)\}, \tag{6}$$

where the symbol $\mathcal{Z}_{\rm orig/dual}$ is simplified to $\mathcal{Z}$. From this expression it is seen that $\mathcal{Z}$ is invariant under the exchange $u_1(K) \leftrightarrow u_1^*(K)$ and $u_{-1}(K) \leftrightarrow u_{-1}^*(K)$, which means self-duality of the partition function. The critical point of a self-dual model is obtained, if it is unique, by the fixed-point condition for these Boltzmann factors, $u_{\pm 1}(K_c) = u_{\pm 1}^*(K_c)$, which yields $K_c = \frac{1}{2}\ln(\sqrt{2}+1)$. It is clear that this transition point is shared by the $2d$ Ising model on the square lattice, the $4d$ Z_2 gauge model on the hypercubic lattice and their higher dimensional generalizations [5].

2.2. Duality of random models and conjecture for the critical point

Let us introduce randomness. To investigate critical points of random systems with the aid of the duality formalism, we will utilize the technique of reference [3] with some modifications.

The random model treated here is a system with Z_2 variables and bimodal randomness. The Hamiltonian is written as

$$H = -J \sum_C \tau_C \prod_{\boldsymbol{x} \in \partial C} S_{\boldsymbol{x}}, \tag{7}$$

where τ_C is a quenched random variable dependent on each element C. τ_C takes the value 1 with probability p and -1 with $1-p$. To treat random systems, we employ the standard replica method. Let us consider the n-replicated system and define the averaged Boltzmann factor x_k for an element C, which corresponds to the configuration $\prod_{\boldsymbol{x}\in\partial C} S_{\boldsymbol{x}} = 1$ in $n-k$ replicas and -1 in k replicas. The explicit form of x_k is

$$x_k(p,K) = pe^{(n-2k)K} + (1-p)e^{-(n-2k)K}. \tag{8}$$

The n-replicated partition function is, after average over randomness, a function of these Boltzmann factors,

$$[\mathcal{Z}^n]_{\rm av} \equiv \mathcal{Z}_n\{x_0(p,K), x_1(p,K), \ldots, x_n(p,K)\}, \tag{9}$$

where $[\ ]_{\rm av}$ means random average.

We also define the dual Boltzmann factor $x_k^*(p,K)$ on the dual lattice. The explicit forms are obtained by the two-component Fourier transformation with the result

$$\begin{aligned} x_{2k}^*(p,K) &= 2^{-n/2}(e^K+e^{-K})^{n-2k}(e^K-e^{-K})^{2k}, \\ x_{2k+1}^*(p,K) &= 2^{-n/2}(2p-1)(e^K+e^{-K})^{n-2k-1}(e^K-e^{-K})^{2k+1}, \end{aligned} \tag{10}$$

where k is a non-negative integer in the range $0 \le 2k < 2k+1 \le n$. The partition function satisfies a generalization of (5),

$$\mathcal{Z}_{n,\rm orig}\{x_0, x_1, \cdots, x_n\} = 2^{\tilde{a}}\mathcal{Z}_{n,\rm dual}\{x_0^*, x_1^*, \cdots, x_n^*\}, \tag{11}$$

where $\tilde{a}$ is an appropriate constant. Now we restrict our attention to the case where the system is self-dual when we remove randomness (e.g. $2d$ $\pm J$ random bond Ising model on the square lattice). Using (11) we can express the self duality of the n-replicated partition function for such a system as

$$\mathcal{Z}_n\{x_0, x_1, \ldots, x_n\} = \mathcal{Z}_n\{x_0^*, x_1^*, \ldots, x_n^*\}, \tag{12}$$

where an overall constant is neglected. Thus self duality is recognized by the fact that $\mathcal{Z}_n$ is invariant when the exchanges $x_k(p,K) \leftrightarrow x_k^*(p,K)$ for all k are performed simultaneously.

It is in general impossible to identify the transition point from the fixed-point condition of the duality relation (12), unlike the non-random case because the fixed-point conditions of all the variables $x_0 = x_0^*, x_1 = x_1^*, \cdots, x_n = x_n^*$ are not satisfied simultaneously. The authors of [3, 4] nevertheless developed an argument leading to a conjecture that the fixed-point condition of the leading Boltzmann factor

$$x_0(p_c, K_c) = x_0^*(p_c, K_c) \tag{13}$$

can well be the most plausible candidate to give the exact transition point of the random system at least on the Nishimori line (NL) [7], $e^{-2K} = (1-p)/p$, where enhanced symmetry simplifies the system properties significantly.

This prediction has been confirmed to be correct in the cases of $n = 1, 2$ and ∞ [3]. It has also been shown that numerical results for the quenched limit $n \to 0$ [8] agree

very well with the conjectured value of $p_c = 0.889972...$ which is the solution to the formula obtained in the $n \to 0$ limit of $x_0 = x_0^*$ on the NL:

$$-p\log_2 p - (1-p)\log_2(1-p) = \frac{1}{2}. \tag{14}$$

This conjecture also leads us to an interesting result that the multicritical points of the $2d$ $\pm J$ random bond Ising model and the $4d$ random Z_2 lattice gauge model (random plaquette gauge model) are located at the same point on the p-K plane [4]. This observation has also been confirmed numerically by a recent study [9].

Determination of the multicritical point is also quite important from the standpoint of the quantum information theory, quantum memory in particular. To be specific, the value $1-p_c$ at the multicritical point of the $2d$ random bond Ising model is equivalent to the accuracy threshold of the $2d$ toric code with perfect measurement [10, 11], which is estimated to be $0.110028\ldots$ from the above discussion. In addition, the value $1-p_c$ at the multicritical point of the $4d$ random gauge model also gives the accuracy threshold of the $4d$ toric code (or $3d$ code with imperfect measurement) [4], which is also determined as $0.110028\ldots$.

3. Z_2 models

With the knowledge of the previous section in mind, we proceed to discussions on the non-self-dual cases with and without randomness.

3.1. Duality for non-self-dual cases

In this subsection, we develop an argument for duality of generic non-self-dual Z_2 models. It is clear that a simple duality relation like (5) is not enough to determine the transition point. Nevertheless we show that, by generalizing the arguments in the previous section, one can still derive a relation between the transition points of a model and its dual.

First let us discuss non-random systems. Consider the product of partition functions of the original and dual models with inverse temperatures K_1 and K_2, respectively. From (5), one finds

$$\begin{aligned} &\mathcal{Z}_{\rm orig}\{u_1(K_1), u_{-1}(K_1)\}\mathcal{Z}_{\rm dual}\{u_1(K_2), u_{-1}(K_2)\} \\ &= \mathcal{Z}_{\rm orig}\{u_1^*(K_2), u_{-1}^*(K_2)\}\mathcal{Z}_{\rm dual}\{u_1^*(K_1), u_{-1}^*(K_1)\}, \end{aligned} \tag{15}$$

which indicates that the product is invariant under the simultaneous exchange $u_{\pm 1}(K_1) \leftrightarrow u_{\pm 1}^*(K_2)$ and $u_{\pm 1}(K_2) \leftrightarrow u_{\pm 1}^*(K_1)$. Hence, if there is a unique transition point K_{1c} (resp. K_{2c}) in the original (resp. dual) model, the relation between two critical points is expected to be given by

$$u_{\pm 1}(K_{1c})u_{\pm 1}(K_{2c}) = u_{\pm 1}^*(K_{1c})u_{\pm 1}^*(K_{2c}), \tag{16}$$

which is invariant under the transformation above. One can verify that this is equivalent to $e^{-2K_{2c}} = \tanh K_{1c}$, which gives the correct relation between the two transition points.

Now we move on to random systems. In this case, we can express the duality between two random models (11) as follows,

$$\begin{aligned} &\mathcal{Z}_{n,\mathrm{orig}}\{x_0(p_1,K_1),\ldots,x_n(p_1,K_1)\}\mathcal{Z}_{n,\mathrm{dual}}\{x_0(p_2,K_2),\ldots,x_n(p_2,K_2)\} \\ &= \mathcal{Z}_{n,\mathrm{orig}}\{x_0^*(p_2,K_2),\ldots,x_n^*(p_2,K_2)\}\mathcal{Z}_{n,\mathrm{dual}}\{x_0^*(p_1,K_1),\ldots,x_n^*(p_1,K_1)\}, \end{aligned} \tag{17}$$

where p_1, p_2 and K_1, K_2 denote the probability of positive interaction and the inverse temperature for the original/dual models, respectively. Thus the product of the partition functions is invariant under the simultaneous exchange $x_k(p_1,K_1) \leftrightarrow x_k^*(p_2,K_2)$ and $x_k(p_2,K_2) \leftrightarrow x_k^*(p_1,K_1)$ for all k.

The argument developed so far naturally suggests that the relation between the critical points of the original and dual systems is given by the fixed-point condition of the leading Boltzmann factors at least on the NL. Explicitly, this condition reads

$$x_0(p_{1c},K_{1c})x_0(p_{2c},K_{2c}) = x_0^*(p_{1c},K_{1c})x_0^*(p_{2c},K_{2c}), \tag{18}$$

in conjunction with the NL condition

$$e^{-2K_1} = \frac{1-p_1}{p_1}, \quad e^{-2K_2} = \frac{1-p_2}{p_2}. \tag{19}$$

Equation (18) with (19) is written in terms of p_{1c} and p_{2c} as

$$\left({p_{1c}}^{n+1} + (1-p_{1c})^{n+1}\right)\left({p_{2c}}^{n+1} + (1-p_{2c})^{n+1}\right) = 2^{-n}. \tag{20}$$

We expect that the two multicritical points are related by this equation. If the quenched $(n \to 0)$ limit is taken, this yields the relation,

$$H(p_{1c}) + H(p_{2c}) = 1, \tag{21}$$

where

$$H(p) \equiv -p\log_2 p - (1-p)\log_2(1-p). \tag{22}$$

Therefore our main statement for the non-self-dual Z_2 model is the following: the two critical points on the NLs of the mutually-dual systems with quenched randomness are expected to be related by equation (21).

There are many reasons to believe that our conjecture expressed in (20) and (21) is exact. Some of them are given in the rest of this paper. For simplicity the discussion in the following is restricted to the $2d$ random bond Ising models on the mutually-dual lattices such as the hexagonal and triangular lattices, though the conjecture applies quite generally to arbitrary systems described by the Hamiltonian (7).

Before closing the present subsection, we explain an explicit representation of the dual random models [2, 3], which is necessary in the following discussions. The ratios of dual Boltzmann factors (10) to x_0^* are

$$\begin{aligned} x_{2k}^*/x_0^* &= \tanh^{2k} K, \\ x_{2k+1}^*/x_0^* &= (2p-1)\tanh^{2k+1} K. \end{aligned} \tag{23}$$

These ratios of Boltzmann factors are realized by a system with the following explicit Boltzmann factors written in terms of Ising spin variables

$$A\exp\left(\tilde{K}(S^{(1)} + S^{(2)} + \ldots + S^{(n)}) + \tilde{K}_p S^{(1)}S^{(2)}\ldots S^{(n)}\right), \tag{24}$$

where $S^{(k)}$ is the product of Ising spin variables in the kth replica. For example, in the case of the usual nearest neighbour interactions, $S^{(k)}$ stands for $S_i^{(k)}S_j^{(k)}$. $\tilde{K}$ and $\tilde{K}_p$ are defined by

$$e^{-2\tilde{K}} \equiv \tanh K, \quad e^{-2\tilde{K}_p} \equiv 2p-1 (\equiv \tanh K_p). \tag{25}$$

From the expression (24), the dual of the $\pm J$ random bond Ising model can be interpreted as the model with the non-random Ising interaction in each replica and the interaction between replicas. If the condition of the NL, $\tilde{K} = \tilde{K}_p$, is imposed, this turns to

$$A \exp\left(\tilde{K}(S^{(1)} + S^{(2)} + \ldots + S^{(n)} + S^{(1)}S^{(2)} \ldots S^{(n)})\right). \tag{26}$$

3.2. Verification for $n = 1$

Let us first show that the relation (18) gives the exact answer even without the NL condition when $n = 1$. Equation (18) is, for $n = 1$,

$$(p_c e^{K_{1c}} + (1-p_{1c})e^{-K_{1c}})(p_{2c}e^{K_{2c}} + (1-p_{2c})e^{-K_{2c}}) = \frac{1}{2}(e^{K_{1c}} + e^{-K_{1c}})(e^{K_{2c}} + e^{-K_{2c}}). \tag{27}$$

¿From the expression (24), it is found that the $\pm J$ random bond Ising model for $n = 1$, averaged over randomness, is regarded as a non-random Ising model with coupling $\tilde{K} + \tilde{K}_p$ on the dual lattice. For example, the random bond Ising model on the $2d$ triangular lattice with parameters p_1 and K_1 is equivalent to the non-random Ising model on the hexagonal lattice, whose inverse temperature $\hat{K}_1$ is given by

$$\hat{K}_1 = \tilde{K}_1 + \tilde{K}_{p1}, \tag{28}$$

where $\tilde{K}_{p1}$ is defined by the second expression of (25) with p replaced by $p1$. Conversely, the random bond Ising model on the $2d$ hexagonal lattice with parameters p_2 and K_2 corresponds to the non-random Ising model on the triangular lattice, whose inverse temperature $\hat{K}_2$ is

$$\hat{K}_2 = \tilde{K}_2 + \tilde{K}_{p2}. \tag{29}$$

As is well-known, the non-random Ising models on the $2d$ triangular and the hexagonal lattices are mutually-dual, and two critical inverse temperatures satisfy [12]

$$e^{-2\hat{K}_{1c}} = \tanh \hat{K}_{2c}. \tag{30}$$

Using equation (25) we can confirm the equivalence between equations (27) and (30). Note that the condition of the NL is not used, and this equivalence holds everywhere on the phase boundary.

3.3. Verification for $n = 2$

Next we discuss the two-replica case. Equation (18) for $n = 2$ is

$$(p_{1c}e^{2K_{1c}} + (1-p_{1c})e^{-2K_{1c}})(p_{2c}e^{2K_{2c}} + (1-p_{2c})e^{-2K_{2c}}) = \frac{1}{4}(e^{K_{1c}} + e^{-K_{1c}})^2(e^{K_{2c}} + e^{-K_{2c}})^2. \tag{31}$$

In the rest of this subsection, we restrict our attention onto the NL.

Due to the dual representation (26) for $n = 2$, we know that the random bond Ising model is equivalent to the non-random four-state Potts model on the dual lattice; four states are constructed by the direct product of Ising factors in two replicas $S^{(1)} = \pm 1, S^{(2)} = \pm 1$, and the dual Boltzmann factor is $Ae^{3\tilde{K}}$ for $S^{(1)} = S^{(2)} = 1$ and $Ae^{-\tilde{K}}$ otherwise. Therefore the system corresponds to the four-state Potts model on the dual lattice with coupling $2\tilde{K}$, namely,

$$\beta H = -2\tilde{K}\left(2\delta_{S^{(1)},1}\delta_{S^{(2)},1} - 1\right) - \tilde{K}. \tag{32}$$

For example, the random bond Ising model on the triangular lattice is equivalent to the non-random four-state Potts model on the hexagonal lattice and vice versa. Here we denote the parameters of the random bond Ising model on the triangular lattice by p_1, K_1 and that on the hexagonal lattice by p_2, K_2. Then the inverse temperatures of the corresponding four-state Potts models $\hat{K}_1, \hat{K}_2$ are given by

$$\hat{K}_1 = 2\tilde{K}_1, \qquad \hat{K}_2 = 2\tilde{K}_2, \tag{33}$$

as mentioned above. The dual coupling $\tilde{K}_1$ (or $\tilde{K}_2$) is defined by equation (25).

The inverse critical temperatures of the four-state Potts models on mutually-dual lattices satisfy [12]

$$e^{-2\hat{K}_{2c}} = \frac{e^{\hat{K}_{1c}} - e^{-\hat{K}_{1c}}}{e^{\hat{K}_{1c}} + 3e^{-\hat{K}_{1c}}}. \tag{34}$$

It is straightforward to show that equation (31) is equivalent to equation (34) under the NL condition (19).

3.4. The limit $n \to \infty$

Next is an argument for the $n \to \infty$ limit. The partition function for arbitrary n is expressed as

$$\mathcal{Z}_n = [\mathcal{Z}^n]_{\rm av} = [e^{-n\beta N f(K)}]_{\rm av}, \tag{35}$$

where N is the number of sites and $f(K)$ is the free energy for the inverse temperature K. If the replica number n is taken to be quite large, the averaged partition function is dominated by the configuration of the smallest value of $f(K)$, which is realized by the perfect ferromagnetic bond configuration and its gauge equivalents. Using this argument $\mathcal{Z}_n$ can be approximated as

$$\mathcal{Z}_n \approx e^{-n\beta N f_0(K)}, \tag{36}$$

where $f_0(K)$ is the free energy of the non-random ferromagnetic system. Therefore the random bond Ising system for $n \to \infty$ is interpreted as the non-random ferromagnetic Ising system (and its critical point is denoted by K_{1c}).

We also consider the random bond Ising model on the dual lattice, and for $n \to \infty$ we can regard it as the non-random Ising model with the critical point K_{2c}. These two critical points are related by the well-known equation,

$$e^{-2K_{2c}} = \tanh K_{1c}. \tag{37}$$

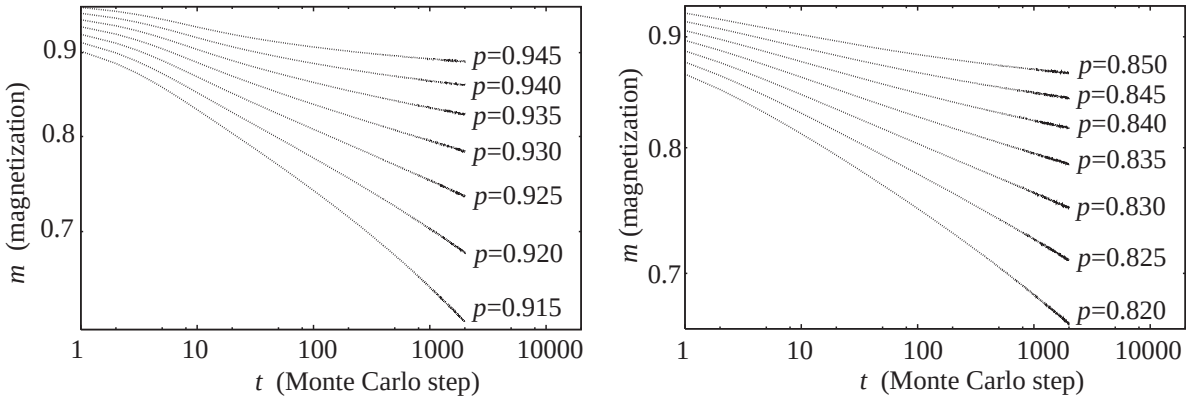

Figure 1. Relaxation of magnetization near the multicritical points of the random bond Ising model on the hexagonal (left) and the triangular (right) lattices from NER analysis. We prepared $L \times L (L = 10^3)$ spins and averaged the results over 200 samples. The system at criticality is expected to yield a straight line in this log-log plot.

If we consider the $n \to \infty$ limit of equation (18) combined with the NL condition (19), we obtain

$$e^{K_{1c}+K_{2c}} = \frac{(e^{K_{1c}} + e^{-K_{1c}})(e^{K_{2c}} + e^{-K_{2c}})}{2}, \tag{38}$$

which is equivalent to equation (37).

3.5. Numerical evidence

We have checked the relation (21) in the quenched limit numerically. We executed Monte Carlo simulations for the $2d$ random bond Ising models on the hexagonal and the triangular lattices, a dual pair. To observe the criticality, we make use of the non-equilibrium relaxation (NER) method [13], which yields the power-law behaviour of decreasing magnetization with Monte Carlo steps on criticality. For this method we prepare all-up spins as the initial state and let them relax in each Monte Carlo step. In order to identify the multicritical point, we choose the parameters of the system to be on the NL and vary the parameter p (and K is also varied accordingly).

The results are shown in figure 1. From these results, the locations of the multicritical points are estimated at $p_{1c} = 0.930(5)$ (or $0.347 < H(p_{1c}) < 0.384$) for the hexagonal lattice and at $p_{2c} = 0.835(5)$ (or $0.634 < H(p_{2c}) < 0.658$) for the triangular lattice. The two H's sum up to $0.981 < H(p_{1c}) + H(p_{2c}) < 1.042$, a consistent result with our conjecture in equation (21).

3.6. Random models in $3d$

As already noted in section 2, the random bond Ising model and the random Z_2 lattice gauge model on the $3d$ cubic lattice are mutually dual. Thus we expect that the

multicritical points of these two models will satisfy equation (21). This conjecture can be confirmed by numerical simulations of these $3d$ systems.

In fact the multicritical points of these two models have already been estimated. For the $3d$ random bond Ising model the multicritical point is estimated to be $p_{1c} = 0.7673(3)$ [14] which yields $H(p_{1c}) \approx 0.783$. For the $3d$ random gauge model it is $p_{2c} \approx 0.967$ [15], giving $H(p_{2c}) \approx 0.209$. From these results the sum $H(p_{1c}) + H(p_{2c})$ is about 0.992, a reasonable value in view of our expectation (21).

From the viewpoint of the quantum information theory, the accuracy threshold of the $2d$ toric code with imperfect measurement can be determined by the value $1 - p_c$ at the multicritical point of the $3d$ random gauge model [10, 11]. Therefore it is an advantage of the present analysis that the duality gives an alternative way to determine the accuracy threshold when we utilize the result of the $3d$ random bond Ising model. From the result of the direct numerical analysis of the $3d$ random gauge model [15] and the duality argument combined with the numerical result of the $3d$ random bond Ising model [14], the accuracy threshold is estimated to be about $0.03 \sim 0.035$.

3.7. 2d anisotropic random bond Ising model and self duality

In this and the next subsections we discuss the $2d$ random bond Ising model with anisotropic disorder. This model is *not* a non-self-dual model, but the structure of the problem is similar to the mutually-dual case. We show that the duality formalism can be applied to this system and gives a conjecture for the critical points. It should also be noted that the critical point of this model gives the accuracy threshold of the $1d$ quantum repetition code with imperfect measurement [11].

First we review the duality of the anisotropic non-random Ising model on the $2d$ square lattice. The Hamiltonian is

$$H = -J_h \sum_{\langle ij \rangle \in C_h} S_i S_j - J_v \sum_{\langle ij \rangle \in C_v} S_i S_j, \tag{39}$$

where J_h and J_v are uniform coupling constants. The symbol C_h denotes the set of horizontal bonds and C_v is for vertical ones. The partition function is

$$\mathcal{Z} = \sum_{\{S_i = \pm 1\}} \prod_{\langle ij \rangle \in C_h} u_{S_{ij}}(K_h) \prod_{\langle ij \rangle \in C_v} u_{S_{ij}}(K_v), \tag{40}$$

where $K_{h,v} \equiv \beta J_{h,v} = J_{h,v}/kT$ and $S_{ij} \equiv S_i S_j$. The symbol u is the Boltzmann factor for the bond between i and j,

$$u_{S_{ij}}(K_{h,v}) \equiv \exp(K_{h,v} S_{ij}). \tag{41}$$

Next we define the dual Boltzmann factor by the binary Fourier transformation,

$$u^*_{\pm 1}(K_{h,v}) \equiv \frac{u_1(K_{h,v}) \pm u_{-1}(K_{h,v})}{\sqrt{2}} = \frac{e^{K_{h,v}} \pm e^{-K_{h,v}}}{\sqrt{2}}. \tag{42}$$

Using these Boltzmann factors, we can express the duality of partition function as

$$\mathcal{Z}\{u_1(K_h), u_{-1}(K_h), u_1(K_v), u_{-1}(K_v)\} = \mathcal{Z}\{u^*_1(K_v), u^*_{-1}(K_v), u^*_1(K_h), u^*_{-1}(K_h)\}. \tag{43}$$

In this expression the Boltzmann factors of the vertical and horizontal bonds are exchanged because a vertical bond is mapped to a horizontal bond on the dual lattice and vice versa.

It is obvious that the partition function is invariant under the exchange $u_{\pm 1}(K_h) \leftrightarrow u^*_{\pm 1}(K_v)$ and $u_{\pm 1}(K_v) \leftrightarrow u^*_{\pm 1}(K_h)$, which is similar to the case of mutually-dual non-random systems. The critical points are determined by the equation,

$$u_{\pm 1}(K_h)u_{\pm 1}(K_v) = u^*_{\pm 1}(K_v)u^*_{\pm 1}(K_h), \tag{44}$$

which yields $e^{-2K_v} = \tanh K_h$.

Next we study the system with randomness. The Hamiltonian is

$$H = -J_v \sum_{\langle ij \rangle \in C_v} \tau^v_{ij} S_i S_j - J_h \sum_{\langle ij \rangle \in C_h} \tau^h_{ij} S_i S_j, \tag{45}$$

where $\tau^{v,h}$ are random variables which depend on each bond and obey the probability distribution,

$$P(\tau^{h,v}_{ij}) = p_{h,v}\delta(\tau^{h,v}_{ij} - 1) + (1 - p_{h,v})\delta(\tau^{h,v}_{ij} + 1), \tag{46}$$

for horizontal(h) or vertical(v) bond. The averaged partition function is a function of averaged Boltzmann factors $x_k(p_h, K_h)$ and $x_k(p_v, K_v)$,

$$[\mathcal{Z}^n]_{\rm av} \equiv \mathcal{Z}_n\{x_0(p_h, K_h), \ldots, x_n(p_h, K_h), x_0(p_v, K_v), \ldots, x_n(p_v, K_v)\}. \tag{47}$$

We also define the dual averaged Boltzmann factors in the same way as in section 2,

$$\begin{aligned} x^*_{2k}(p_{h,v}, K_{h,v}) &= 2^{-n/2}(e^{K_{h,v}} + e^{-K_{h,v}})^{n-2k}(e^{K_{h,v}} - e^{-K_{h,v}})^{2k}, \\ x^*_{2k+1}(p_{h,v}, K_{h,v}) &= 2^{-n/2}(2p_{h,v} - 1)(e^{K_{h,v}} + e^{-K_{h,v}})^{n-2k-1}(e^{K_{h,v}} - e^{-K_{h,v}})^{2k+1}. \end{aligned} \tag{48}$$

Using x_k and x^*_k, we can express the duality of the n-replicated partition function,

$$\begin{aligned} &\mathcal{Z}_n\{x_0(p_h, K_h), \ldots, x_n(p_h, K_h), x_0(p_v, K_v), \ldots, x_n(p_v, K_v)\} \\ &= \mathcal{Z}_n\{x^*_0(p_v, K_v), \ldots, x^*_n(p_v, K_v), x^*_0(p_h, K_h), \ldots, x^*_n(p_h, K_h)\}, \end{aligned} \tag{49}$$

which is invariant under the simultaneous exchange $x_k(p_h, K_h) \leftrightarrow x^*_k(p_v, K_v)$ and $x_k(p_v, K_v) \leftrightarrow x^*_k(p_h, K_h)$ for all k. The Boltzmann factors of the vertical and horizontal bonds should be exchanged as in the non-random case.

From the argument above, we make a conjecture for the critical points on the NLs,

$$x_0(p_h, K_h)x_0(p_v, K_v) = x^*_0(p_h, K_h)x^*_0(p_v, K_v), \tag{50}$$

from the analogy with the non-random case (44) or the non-self-dual random case (18). In this system, the NLs are defined for the horizontal and vertical bonds, respectively, by

$$e^{-2K^h} = \frac{1 - p_h}{p_h}, \quad e^{-2K^v} = \frac{1 - p_v}{p_v}. \tag{51}$$

We should note that the original system has four parameters p_h, K_h, p_v, K_v and the two conditions of the NLs reduce the number of independent variables to two. Consequently, the conjecture (50) combined with the conditions of the NLs (51) does not fix the multicritical point on the phase diagram but determines the location of the "critical line". This fact is favourable for numerical verification of our conjecture because one parameter can be chosen freely even on criticality and on the NLs.

3.8. Verification for anisotropic system

We verify the validity of the conjecture in equation (50). For this purpose the following property is useful; the form of equation (50) is completely the same as equation (18) when the parameters are replaced as

$$\{p_1, K_1, p_2, K_2\} \to \{p_h, K_h, p_v, K_v\}. \tag{52}$$

Using this correspondence we can check the conjecture (50) by the same argument as in mutually-dual systems with $n = 1, 2$ and $n \to \infty$.

- $n = 1$: The anisotropic random bond Ising model is equivalent to the anisotropic non-random Ising model, and equation (30) turns to the critical condition for the anisotropic non-random model when the replacement (52) is used.
- $n = 2$: The system is equivalent to the anisotropic non-random four-state Potts model. Using the replacement (52), equation (34) becomes the critical condition for the anisotropic Potts model.
- $n \to \infty$: The system is equivalent to the anisotropic non-random Ising model and the critical point satisfies equation (37) using the correspondence (52).

  If we consider the quenched limit $n \to 0$, we obtain the relation,

$$H(p_h) + H(p_v) = 1, \tag{53}$$

by the same argument as in mutually-dual systems (21). Here we take p_h, p_v as the two independent variables and eliminate $K_{h,v}$ using the conditions of the NLs. We expect that p_h and p_v will satisfy equation (53) on criticality.

We have checked equation (53) numerically. The result is shown in figure 2. The locations of the critical points expected from equation (53) are in reasonable agreement with numerical results. Using the NER with ordered and disordered initial configurations, we also checked that the result for p_v very close to 1 (or $p_h \sim 1$) does not show discrepancy from the conjecture, which is not shown in the figure because the critical point could not be determined with sufficient precision due to rigid spin domains along one axis. Thus we conclude that our conjecture is consistent with the numerical result for the anisotropic system as well.

We are allowed to interpret (53) as the accuracy threshold of the $1d$ quantum repetition code, given by the value of $1 - p_h$ under the imperfect measurement rate $1 - p_v$.

4. $2d$ Z_q model

In this section we consider a generalization of the formalism to a multi-valued spin model, the random Z_q model. The Z_q model has q states per site. In the non-random case, this system in two dimensions has both aspects of the ferromagnetic Ising model (i.e. with ferromagnetic order) and the XY model (i.e. with the Kosterlitz-Thouless (KT) order) for q not too small [16, 17]. The authors of reference [17] showed that this

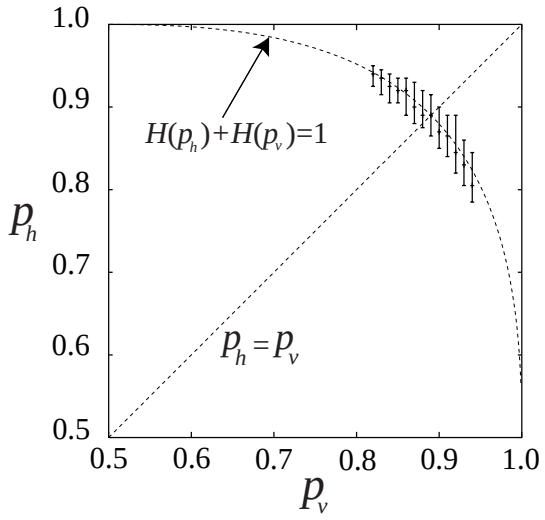

Figure 2. Results by the NER for the anisotropic random bond Ising model. We simulated systems with up to 500×500 spins under all-up initial configuration. The results are averaged over 200 samples. For executing numerical calculation, we fixed p_v and scanned p_h in order to search the critical point. The critical points obtained are shown with error bars. The curve $H(p_h) + H(p_v) = 1$ lies within the error bars for all p_v.

model has three phases for $q \geq 5$, disordered, KT-like, and ferromagnetic, if there exists a KT phase in the continuum limit $q \to \infty$. Duality has been used to relate the two transition points.

The objective of the present section is to generalize the duality arguments in the previous section and apply them to the randomized version of the Z_q model. As far as the authors know, this type of models have not been studied analytically in detail in the literature.

4.1. Duality and analysis of the transition points

We consider the random chiral Z_q model, for which the partition function is written in the form,

$$\mathcal{Z} = \sum_{\{k\}} \prod_{\langle ij \rangle} e^{V_K(k_i - k_j - l_{ij})}, \tag{54}$$

where $V_K(k)$ is the interaction which satisfies cyclic condition, $V_K(k+q) = V_K(k)$. Here i, j are the site indexes, k is the state variable which takes q values, $0, 1, \cdots, q-1$, with $q \in \mathbb{Z}_+ = \{1, 2, \cdots\}$ and l is the random variable on each bond which also takes q values, $0, 1, \cdots, q-1$. The probability that a random variable l on a bond takes the value l is denoted by p_l ($\sum_l p_l = 1$). A collection of them is denoted by $\{p\}(= \{p_0, p_1 \ldots p_{q-1}\})$ in the sequel. The summation $\sum_{\{k\}}$ is taken over state variables on all sites and the product $\prod_{\langle ij \rangle}$ is taken over all bonds. Only nearest neighbour interactions are assumed to exist on the square lattice in the present section.

Our discussions are mostly for the Z_q version of the Villain model [18], for which the Boltzmann factor is given by

$$e^{V_K(k)} = \sum_{m=-\infty}^{\infty} e^{-\frac{K}{2}(\frac{2\pi}{q}k-2\pi m)^2} \left(= \frac{1}{\sqrt{2\pi K}} \sum_{l\in\mathbb{Z}} e^{-l^2/(2K)+2\pi i l k/q} \right). \tag{55}$$

We generalize the duality arguments to the random Z_q model. Let us average the n-replicated partition function. The resulting function $\mathcal{Z}_n$ can be written in terms of the q^n variables (local Boltzmann factors for neighbouring spin pairs),

$$\chi_{k_1,\cdots,k_n}(\{p\},K) \equiv \sum_l p_l e^{V_K(l+k_1)+\cdots+V_K(l+k_n)}, \tag{56}$$

where $k_i = 0,1,\cdots,q-1$ $(i = 1,2,\cdots,n)$. The variable $\chi_{k_1,\cdots,k_n}(\{p\},K)$ is the generalization of the Boltzmann factor $x_k(p,K)$ which appeared in section 2. If we set $q=2$ and $V_K(0)=K, V_K(1)=-K$, we obtain the relation between the present and previous Boltzmann factors,

$$\chi_{k_1,\cdots,k_n}(\{p\},K) = x_k(p,K), \quad \text{where} \quad k=\sum_{m=1}^{n} k_m \quad (k_m=0,1). \tag{57}$$

We can define the dual model, for which the partition function can be written in terms of the dual variables. They are defined by q-value Fourier transformations [6]

$$\chi^*_{k_1,\cdots,k_n}(\{p\},K) \equiv q^{-n/2} \sum_l p_l \sum_{m_1} \omega^{k_1 m_1} e^{V_K(m_1+l)} \cdots \sum_{m_n} \omega^{k_n m_n} e^{V_K(m_n+l)}, \tag{58}$$

where $\omega = e^{2\pi i/q}$. A remark is in order. The variables $\chi_{k_1,\cdots,k_n}$ are already defined at the end of section 2 of reference [3] in a different form from the ones given above. The expressions (74)-(79) in [3] are not in general valid and should be replaced by (57) and (58).

Now we define

$$\begin{aligned} \chi_{\rm O}(\{p\},K) &\equiv \chi_{0,\cdots,0}(\{p\},K) = \sum_{l=0}^{q-1} p_l e^{nV_K(l)}, \\ \chi^*_{\rm O}(\{p\},K) &\equiv \chi^*_{0,\cdots,0}(\{p\},K) = q^{-n/2} \sum_{l=0}^{q-1} p_l \left(\sum_{k=0}^{q-1} e^{V_K(k+l)} \right)^n, \end{aligned} \tag{59}$$

and apply the same arguments as in the previous section to identification of transition points of the Z_q model. If uniqueness of the critical point is assumed, generalization of (13) gives

$$\chi_{\rm O}(\{p\},K) = \chi^*_{\rm O}(\{p\},K). \tag{60}$$

Remember that the non-random model does not have a unique transition point for $q \geq 5$ [17]. Rather, two transition points are related to each other by the duality relation. Therefore let us consider the case where there exist two transition points for the random case. If there are two critical points K_1, K_2 for a given $\{p\}$, they may satisfy

$$\chi_{\rm O}(\{p\},K_1)\chi_{\rm O}(\{p\},K_2) = \chi^*_{\rm O}(\{p\},K_1)\chi^*_{\rm O}(\{p\},K_2). \tag{61}$$

Let us take the quenched ($n \to 0$) limit of these relations. When uniqueness is assumed ($K_1 = K_2$), the condition reads

$$-\sum_{l=0}^{q-1} p_l \log_q \left(\frac{e^{V_K(l)}}{\sum_{k=0}^{q-1} e^{V_K(k+l)}} \right) = \frac{1}{2}. \tag{62}$$

If there are two critical points, this is replaced by

$$-\sum_{l=0}^{q-1} p_l \log_q \left(\frac{e^{V_{K_1}(l)}}{\sum_{k=0}^{q-1} e^{V_{K_1}(k+l)}} \cdot \frac{e^{V_{K_2}(l)}}{\sum_{k=0}^{q-1} e^{V_{K_2}(k+l)}} \right) = 1. \tag{63}$$

If we consider the non-random case, where $p_0 = 1, p_l = 0 (l \neq 0)$, $\chi_{\rm O}$ and $\chi_{\rm O}^*$ read

$$\chi_{\rm O}(\{1, 0 \ldots\}, K) = \sum_{m \in \mathbb{Z}} e^{-\frac{K}{2}(2\pi m)^2}, \tag{64}$$

$$\chi_{\rm O}^*(\{1, 0 \ldots\}, K) = q^{-1/2} \sum_{k=0}^{q-1} \sum_{m \in \mathbb{Z}} e^{-\frac{K}{2}(\frac{2\pi}{q} k - 2\pi m)^2} = \frac{1}{\sqrt{2\pi K q}} \sum_{m \in \mathbb{Z}} e^{-\frac{q^2 m^2}{2K}}. \tag{65}$$

If there are two critical points, equation (61) gives the relation between them,

$$K_1 K_2 = \frac{q^2}{4\pi^2}. \tag{66}$$

This agrees with the correct relation [17]. However, it is in general not expected that the condition (61) determines the whole shape of the phase boundary in the phase diagram of the random Z_q model. Restriction to the NL, where enhanced symmetry helps us to predict various exact results, is more likely to lead to reliable results.

4.2. Duality on the NL

Consider a specific choice of p_l,

$$p_l(K_p) = \frac{e^{V_{K_p}(l)}}{\sum_{k=0}^{q-1} e^{V_{K_p}(k)}}, \tag{67}$$

which enables us to obtain exact results using gauge symmetry under the NL condition [19],

$$K = K_p. \tag{68}$$

The above set $\{p_l\}$ on the NL is denoted by $\{p_{\rm NL}\}$. As in the previous section, we try to identify the relation between two transition points on the NL using the condition (61).

Let therefore $\chi_{\rm O}^{\rm NL}(K), \chi_{\rm O}^{*\,\rm NL}(K)$ denote the variables on the NL,

$$\begin{aligned} \chi_{\rm O}^{\rm NL}(K) \ &\equiv \chi_{\rm O}(\{p_{\rm NL}\}, K) = \sum_{l=0}^{q-1} p_l(K) e^{nV_K(l)}, \\ \chi_{\rm O}^{*\,\rm NL}(K) &\equiv \chi_{\rm O}^*(\{p_{\rm NL}\}, K) = q^{-n/2} \sum_{l=0}^{q-1} p_l(K) \left(\sum_{k=0}^{q-1} e^{V_K(k+l)} \right)^n. \end{aligned} \tag{69}$$

Our conjecture for the replicated systems is the following. If uniqueness of the critical point is assumed, the location of it is determined by

$$\chi_{\rm O}^{\rm NL}(K) = \chi_{\rm O}^{*\,\rm NL}(K). \tag{70}$$

If there are two critical points K_1, K_2 on the NL, they satisfy

$$\chi_{\rm O}^{\rm NL}(K_1)\chi_{\rm O}^{\rm NL}(K_2) = \chi_{\rm O}^{*\,\rm NL}(K_1)\chi_{\rm O}^{*\,\rm NL}(K_2). \tag{71}$$

We expect these relations to hold even in the quenched ($n \to 0$) limit. In this limit, the condition reads, corresponding to equation (70),

$$-\sum_{l=0}^{q-1} p_l(K)\log_q p_l(K) = \frac{1}{2}, \tag{72}$$

and to equation (71)

$$-\sum_{l=0}^{q-1}\left(p_l(K_1)\log_q p_l(K_1) + p_l(K_2)\log_q p_l(K_2)\right) = 1. \tag{73}$$

It is interesting to note that these conditions are written in terms of the entropy function. This fact is suggestive of some underlying structure behind the scene, but we do not have a clear interpretation of this fact at the moment.

It is convenient to define

$$F_K(x) \equiv \sum_{m=-\infty}^{\infty} e^{-\frac{K}{2}(x-2\pi m)^2} \tag{74}$$

to facilitate more compact expressions for duality relation. Clearly, we have

$$e^{V_K(k)} = F_K\left(\frac{2\pi k}{q}\right). \tag{75}$$

We also see

$$\begin{aligned}\sum_{k=0}^{q-1} e^{V_K(k+l)} &= \sum_{k=0}^{q-1}\sum_{m\in\mathbb{Z}} e^{-\frac{K}{2}\left(\frac{2\pi}{q}(k+l)-2\pi m\right)^2} = \sum_{k=0}^{q-1}\frac{1}{\sqrt{2\pi K}}\sum_{l'} e^{-\frac{(l')^2}{2K}+il'\frac{2\pi(k+l)}{q}} \\ &= \frac{q}{\sqrt{2\pi K}}\sum_{m\in\mathbb{Z}} e^{-\frac{m^2q^2}{2K}} = \sum_{m\in\mathbb{Z}} e^{-\frac{2\pi^2 K m^2}{q^2}} = F_{K/q^2}(0),\end{aligned} \tag{76}$$

so that we have

$$p_l(K) = \frac{F_K\left(\frac{2\pi k}{q}\right)}{F_{K/q^2}(0)}. \tag{77}$$

Hence equations (70)–(73) can be rewritten in terms of $F_K(x)$. It should be noticed that the function $F_K(x)$ can be written in terms of ϑ_3, a Theta function:

$$F_K(x) = e^{-\frac{K}{2}x^2}\sum_{m\in\mathbb{Z}}(e^{-2\pi^2 K})^{m^2}(e^{\pi K x})^{2m} = e^{-\frac{K}{2}x^2}\sum_{m\in\mathbb{Z}} Q^{m^2}z^{2m} = e^{-\frac{K}{2}x^2}\vartheta_3(u,Q), \tag{78}$$

where $z = e^{\pi K x} = e^{ui}$ ($u = -i\pi K x$) and $Q = e^{-2\pi^2 K}$. This expression is useful for numerical evaluations of the transition points on the computer with special functions software implemented.

Now we would like to discuss the plausibility of our conjectures (70) - (73). The $q = 2$ and the $q = 3$ cases are equivalent to the Ising model and the three state

Potts model, respectively, both of which have been discussed in detail in [3]. These are favourable facts to support our conjectures. Unfortunate aspect about the Z_q model is the difficultly of the analysis of the replicated model even for $n = 1$. Hence we do not give such evidence as we did for the Z_2 case. Instead, the limit $q \gg 1$ is discussed in the next subsection.

4.3. The limit $q \to \infty$

When q is large and the transition point is unique, it is reasonably expected that the transition point is of order $O(q)$. In fact, if we set $K = \gamma q$ and suppose q is large, we find a consistent solution as follows. The functions appearing in (77) behave asymptotically as

$$F_{K/q^2}(0) \quad = \frac{q}{\sqrt{2\pi K}} \sum_{m\in\mathbb{Z}} e^{-\frac{m^2q^2}{2K}} \sim \frac{q}{\sqrt{2\pi K}}, \tag{79}$$

$$F_{\gamma q}\left(\frac{2\pi l}{q}\right) = \sum_{m\in\mathbb{Z}} e^{-\frac{\gamma}{2}(\frac{2\pi}{\sqrt{q}}l - 2\pi\sqrt{q}m)^2} \sim e^{-\frac{\gamma}{2}(\frac{2\pi l}{\sqrt{q}})^2}. \tag{80}$$

Substitution of these expressions into (72) leads to

$$\sqrt{2\pi\gamma}\sum_l \frac{\gamma}{2\sqrt{q}}\left(\frac{2\pi l}{\sqrt{q}}\right)^2 e^{-\frac{\gamma}{2}(\frac{2\pi l}{\sqrt{q}})^2} = \frac{1}{2}\log(2\pi\gamma). \tag{81}$$

If we approximate the summation by an integral, which should be valid for large q, we find

$$\text{LHS} \sim \sqrt{2\pi\gamma}\int_{-\infty}^{\infty} dy \frac{\gamma}{2}(2\pi y)^2 e^{-\frac{\gamma}{2}(2\pi y)^2} = \frac{1}{2}, \tag{82}$$

where we set $y = l/\sqrt{q}$. We therefore have

$$\log(2\pi\gamma) = 1, \tag{83}$$

implying $\gamma = e/2\pi$ with e being the base of natural logarithm. Hence, when uniqueness of the critical point is assumed, the asymptotic location is

$$K = \frac{qe}{2\pi}. \tag{84}$$

When there are two critical points K_1, K_2, the above argument should be modified to some extent. If we assume that both of K_1 and K_2 are of order $O(q)$, we may set $K_i = \gamma_i q$ $(i = 1, 2)$ and apply the same procedure as above. The result is that K_1 and K_2 should be related through

$$K_1 K_2 = \frac{q^2 e^2}{4\pi^2}. \tag{85}$$

However, for the non-random case, it is known that the two transition points are not of order $O(q)$; one is of order $O(q^2)$ and the other $O(1)$ [17]. Since this is expected to persist for the random case (see subsection 4.4 below), equation (85) may not necessarily capture the true asymptotics of the transition points.

4.4. Structure of the phase diagram

In this subsection we discuss the phase structure of the random Z_q model. For the case of the non-random model, it was shown that, when q is sufficiently large, the simple Ising-like two phase picture (with ferromagnetic and paramagnetic phases) is not possible [17]. The authors of [17] showed that the transition point determined by the duality in the case of only two phases is inconsistent with a kind of Griffith's inequality. The discussions were based on the assumption of the existence of the KT transition in the continuous model $q \to \infty$.

We would like to address the issue of the full phase structure of the random Z_q model but let us restrict our main interest on the NL for a moment. Basically it is expected that the phase transitions on the NL are of a similar nature to the non-random case. There are three phases; disordered, KT and ferromagnetic. This conclusion may be drawn by using the same arguments as for the non-random case. There are, however, still some debates about the existence of the KT phase when randomness is introduced in the model [20, 21, 22]. Hence, in this section, we give a different argument supporting the three-phase picture without assuming the existence of the KT transition in the continuous model. A crucial point in our arguments is the proof of existence and non-existence of the ferromagnetic phase in appropriate parameter regions when $q \gg 1$.

First we show the existence of a ferromagnetic phase near the limit $K, K_p \to \infty$ (the ground state of the non-random system) following [23, 24]. Let us first consider the non-random case. The order parameter is bounded as

$$\begin{aligned}\langle e^{2\pi i \frac{k_j}{q}} \rangle_K &= \sum_{k=0}^{q-1} P_k e^{2\pi i \frac{k}{q}} = P_0 + \sum_{k(\neq 0)} P_k e^{2\pi i \frac{k}{q}} \geq P_0 - \sum_{k(\neq 0)} P_k = 1 - 2(1 - P_0) \\ &= 1 - 2\langle N' \rangle_K / N, \end{aligned} \tag{86}$$

where $\langle\ \rangle_K$ means thermal average, P_k is the probability that the spin takes the value k on each site, and N' is the number of sites such that $k \neq 0$. All boundary spins have $k = 0$. As in [23, 24], N' is bounded as

$$N' \leq \sum_{b=4,6,\cdots} \left(\frac{b}{4}\right)^2 \sum_{j=1}^{\nu(b)} X_b^{(j)}, \tag{87}$$

where $X_b^{(j)}$ is 1 if the jth border (separating a domain of sites from those with different k's) of length b occurs in a configuration and 0 otherwise. The symbol $\nu(b)$ stands for the number of possible borders of length b. The average $\langle X_b^{(j)} \rangle_K$ is bounded by, for large q,

$$\langle X_b^{(j)} \rangle_K \leq e^{-\frac{Kb}{2}\left(\frac{2\pi}{q}\right)^2}. \tag{88}$$

Using

$$\nu(b) \leq 4 \cdot 3^b q^b N/(3b), \tag{89}$$

where the factor q^b comes from the possible number of boundary bonds, we find

$$\frac{\langle N' \rangle_K}{N} \leq \frac{1}{12} \sum_{b=4,6,\cdots} b 3^b q^b e^{-\frac{Kb}{2}\left(\frac{2\pi}{q}\right)^2}. \tag{90}$$

Hence if we choose $\kappa \equiv 3qe^{-\frac{K}{2}\left(\frac{2\pi}{q}\right)^2}$ sufficiently small, the order parameter, the left-hand side of (86), is certainly positive. From this follows that the ferromagnetic phase exists when $T < C_1^{(0)}/q^2$ with $C_1^{(0)}$ being some positive constant.

Following [24], we can generalize the above argument to the random case. At least for K and K_p very large, it is possible to prove the existence of a ferromagnetic phase. Let us take a border of spin configurations such that $X_b^{(j)} \neq 0$ and notice that the thermal average $\langle X_b^{(j)} \rangle_K$ is written as

$$\langle X_b^{(j)} \rangle_K = \frac{\sum'_{\{k\}} \prod_{\langle ij \rangle} e^{V_K(k_i - k_j - l_{ij})}}{\sum_{\{k\}} \prod_{\langle ij \rangle} e^{V_K(k_i - k_j - l_{ij})}}, \tag{91}$$

where the sum in the numerator is taken over states in which the jth border of length b occurs. Here l_{ij} is the quenched randomness and occurs with probability $r^{l_{i,j}^2}$ ($r = e^{-2\pi^2 K_p/q^2}$) approximately at each bond. Take $b = 2n$ (n to be distinguished from the replica number) and consider the case where the number of such bonds with $l_{i,j} \neq 0$ is m. When $m = n, n+1, \cdots, 2n$, one restricts the summation in the denominator of (91) to configurations which appear in the numerator and obtains a trivial upper bound $\langle X_{2n}^{(j)} \rangle \leq 1$. On the other hand, when $m = 0, 1, \cdots, n-1$, one finds an upper bound $\langle X_{2n}^{(j)} \rangle \leq \lambda^{2n-m-\sum l_{i,j}^2}$ with $\lambda = e^{-2\pi^2 K/q^2}$, where the summation in the exponent is over the bonds with $l_{i,j} \neq 0$. This is obtained by restricting the summation in the denominator in (91) to configurations with all $k_i = 0$. Hence the configurational average $[\langle X_{2n}^{(j)} \rangle_K]_{\rm av}$ is bounded as

$$[\langle X_{2n}^{(j)} \rangle_K]_{\rm av} \leq \sum_{m=0}^{n-1} \left[\begin{array}{c} 2n \\ m \end{array} \right] \sum_{l_{i,j}} r^{\sum l_{i,j}^2} \lambda^{2n-m-\sum l_{i,j}^2} + \sum_{m=n}^{2n} \left[\begin{array}{c} 2n \\ m \end{array} \right] q^m r^m. \tag{92}$$

When $K \leq K_p$, one has $r/\lambda \leq 1$ so that this can be replaced by

$$[\langle X_{2n}^{(j)} \rangle_K]_{\rm av} \leq \sum_{m=0}^{n-1} \left[\begin{array}{c} 2n \\ m \end{array} \right] q^m r^m \lambda^{2n-2m} + \sum_{m=n}^{2n} \left[\begin{array}{c} 2n \\ m \end{array} \right] q^m r^m. \tag{93}$$

Then, if we choose $K_p/q^2 \geq K/q^2 \gg 1$, by almost the same argument as in the non-random case, $\langle N' \rangle_K / N$ can be made sufficiently small so that the order parameter does not vanish. In particular, there exists a positive constant C_1 such that the order is nonzero when $T < C_1/q^2$ on the NL. The ferromagnetic phase is also expected when $K_p/q^2 \geq K/q^2 \gg 1$ but we need a little tighter estimation to prove it.

On the other hand, to show non-existence of ferromagnetic phase at high temperature, we use the arguments in [25]. We show below

$$\left| \left[\langle e^{2\pi i \frac{k_j}{q}} \rangle_K \right]_{\rm av} \right| \leq \langle e^{2\pi i k_j/q} \rangle_{K_p, \rm nonrandom} \langle e^{2\pi i k_j/q} \rangle_{K, \rm nonrandom}, \tag{94}$$

where the average on the right-hand side is taken for the non-random model at inverse temperatures K_p and K, respectively. To verify this inequality, we first notice

$$\left[\langle e^{2\pi i\frac{k_j}{q}}\rangle_K\right]_{\rm av} = \left[\frac{{\rm Tr}_{\{k\}}\, e^{2\pi i k_j/q}\prod_{\langle j_1,j_2\rangle} e^{V_K(k_{j_1}-k_{j_2}-l_{j_1,j_2})}}{{\rm Tr}_{\{k\}}\prod_{\langle j_1,j_2\rangle} e^{V_K(k_{j_1}-k_{j_2}-l_{j_1,j_2})}}\right]_{\rm av}$$
$$= \sum_{\{l\}}\left(\prod_{\langle j_1,j_2\rangle}\frac{e^{V_{K_p}(l_{j_1,j_2})}}{\sum_{l_1,l_2} e^{V_{K_p}(l_{j_1,j_2})}}\right)\frac{{\rm Tr}_{\{k\}}\, e^{2\pi i k_j/q}\prod_{\langle j_1,j_2\rangle} e^{V_K(k_{j_1}-k_{j_2}-l_{j_1,j_2})}}{{\rm Tr}_{\{k\}}\prod_{\langle j_1,j_2\rangle} e^{V_K(k_{j_1}-k_{j_2}-l_{j_1,j_2})}}. \quad (95)$$

Applying the gauge transformation,

$$k_j \quad \to k_j - \kappa_j, \quad (96)$$
$$l_{j_1,j_2} \to l_{j_1,j_2} - \kappa_{j_1} + \kappa_{j_2}, \quad (97)$$

we find

$$\left[\langle e^{2\pi i\frac{k_j}{q}}\rangle_K\right]_{\rm av} = \sum_{\{l\}} P_{\{l\}}\langle e^{2\pi i\frac{\kappa_j}{q}}\rangle_{K_p}\langle e^{2\pi i\frac{k_j}{q}}\rangle_K, \quad (98)$$

where $P_{\{l\}}$ denotes a certain probability distribution. If we apply a very reasonable inequality that the value of the order parameter of the random system is smaller than its non-random counterpart with the same inverse temperature

$$|\langle e^{2\pi i\kappa_j/q}\rangle_{K_p}||\langle e^{2\pi i k_j/q}\rangle_K| \le \langle e^{2\pi i\kappa_j/q}\rangle_{K_p,{\rm nonrandom}}\langle e^{2\pi i k_j/q}\rangle_{K,{\rm nonrandom}} \quad (99)$$

to each term in (98), we arrive at (94). Unfortunately we have not succeeded in proving the inequality (99) mathematically except for the $q = 2$ case, which was already proved in [26]. However, even in the absence of a formal mathematical proof, equation (99) should certainly be valid. Now, if we let K_1 denote the phase transition point of the non-random model at which the order vanishes, we see that the ferromagnetic phase cannot exist when $K < K_1$ or $K_p < K_1$. In fact K_1 is known to be of order $O(q^2)$ [17] and we conclude that the ferromagnetic phase cannot exist on the NL when $T > C_2/q^2$ with C_2 some positive constant.

Notice that the above arguments imply the existence of (at least one) transition point(s) between ferromagnetic and non-ferromagnetic phases. Now if there is a unique transition point, it is of order $O(1/q)$ [cf (84)]. But this contradicts with the fact that the order vanishes when $T > C_2/q^2$. Therefore we have shown that the simple Ising-like two phase picture is not possible when q is large enough. This conclusion strongly suggests that there are three phases in the full phase diagram. A schematic phase diagram expected from the above arguments is given in figure 3.

As a remark, for q not very large, the situation may be more subtle. Numerical simulation results suggest that something peculiar may be happening for these models (e.g. NER analysis in [27]). Further careful analyses are necessary on this point.

4.5. Maximally Random Case

Let us consider the maximally random case ($K_p = 0$), in which the possibility of a finite temperature phase transition has been discussed in the literature [28]. Here we give an argument against such a possibility using duality.

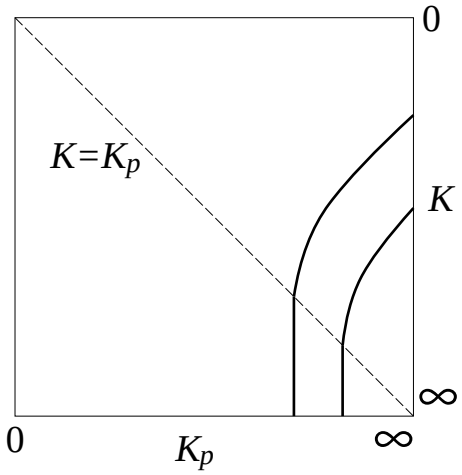

Figure 3. Phase diagram of the random Z_q model. Solid lines show expected phase boundaries.

As in the case of the Ising model [3], the n-replicated system with $K_p = 0$ is related to the $(n-1)$-replicated system on the NL. More precisely, it is easy to prove

$$\mathcal{Z}_n(K, K_p = 0) \propto \mathcal{Z}_{n-1}(K, K_p = K), \tag{100}$$

where $\mathcal{Z}_n(K, K_p)$ stands for the averaged partition function (9). Now let us assume the applicability of this relation down to the $n \to 0$ limit and discuss the phase transitions for the quenched model with $K_p = 0$. Note that the $n \to 0$ limit for $K_p = 0$ is equivalent to taking the $n \to -1$ limit on the NL according to (100).

If we assume that there is a unique transition point, it is determined by the relation (70), which reads

$$\frac{\sum_{l=0}^{q-1} e^{nV_K(l)}}{\left(\sum_{l=0}^{q-1} e^{V_K(l)}\right)^n} = q^{-n+1}. \tag{101}$$

Let us take the $n \to 0$ limit. If we set $K = c/n$, we see

$$\left(\sum_{l=0}^{q-1} e^{V_K(l)}\right)^n \sim e^{V_c(0)}, \tag{102}$$

so that (101) becomes

$$\frac{\sum_{l=0}^{q-1} e^{V_c(l)}}{e^{V_c(0)}} = q. \tag{103}$$

This is nothing but a condition to determine the transition point for the non-random system when there is a unique transition point. Accordingly we know that $c = q/(2\pi)$ by (66) with $K_1 = K_2$ and hence $K = c/n \to \infty$ as $n \to 0$. This means that the phase transition, if it is unique, occurs at $T = 0$. The same discussions can be applied to the case where there are two transition points. In any case the conclusion is that the phase transitions, if there exist, occur only at $T = 0$.

We should remember that our discussions were based on the subtle assumption about the applicability of the relation as $n \to 0$ ($n \to -1$ on the NL) and the assumption that (70) or (71) gives the criticality condition. Hence our conclusion has not been completely rigorously derived. It would be interesting to clarify this point with other methods or arguments.

5. Conclusion and discussion

In this paper, we have generalized the duality argument combined with the replica method, which was originally applied to the $2d$ random bond Ising model on the square lattice, to a variety of random spin systems. Our main results are the conjectures on the transition points of systems with quenched randomness.

First we considered the random Z_2 models. We have given the conjecture (21) about the relation between the multicritical points of two models which are mutually dual. Besides exact computations for the replicated system with $n = 1, 2$ and ∞, numerical simulations support our conjecture for the $2d$ $\pm J$ random bond Ising model on the triangular/hexagonal lattices, the $3d$ random Ising/gauge models on the cubic lattice and the $2d$ anisotropic random bond Ising model on the square lattice. We think it remarkable that a single theoretical framework makes it possible to derive a series of predictions to be compared favourably with many independent numerical simulations.

Next we treated the random Z_q model. We have shown that there are at least two transition points for sufficiently large q. The most probable scenario is that there are three phases in the phase diagram, paramagnetic, KT-like and ferromagnetic phases. Our arguments, however, do not assume the existence or some specific properties of the KT phase. By applying the duality argument to this model, we have also given the conjectured relation (73) between the two transition points.

An interesting question is why we restrict ourselves to the NL. The fixed-point condition of the leading Boltzmann factor $x_0(K, p) = x_0^*(K, p)$ relates p and K and may give the whole shape of the phase boundary. This is indeed the case for $n = 1$ as well as for $n = 2$ above the multicritical point. We nevertheless have restricted ourselves to the NL in this paper because it is difficult to directly verify the ansatz on the whole part of the phase diagram numerically for many systems with quenched randomness. It should also be kept in mind that the multicritical point is the place where two completely different types of symmetries, invariance under duality and gauge transformation, meet under the present conjecture. This implies that the multicritical point has clearly distinguished symmetry features, which allows us to discuss this point on a different basis from other points of phase transition.

We believe that sufficient analytical and numerical evidence has been accumulated to support the validity of our conjecture. It is an interesting future problem to provide a mathematically rigorous proof.

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Appendix

In this appendix we calculate the prefactor in the duality relation of the partition function (5) in section 2. Here we follow the derivation of the duality relation by Wu and Wang [6].

The partition function of the system described by the Hamiltonian (1) in section 2 is

$$\mathcal{Z} = \sum_{\{\xi_{\boldsymbol{x}}=0,1\}} \prod_{C} U(\xi_C), \tag{104}$$

where

$$U(\xi_C) = \prod_{\boldsymbol{x}\in\partial C} \exp\left(K\xi_{\boldsymbol{x}}\right). \tag{105}$$

$U(\xi_C)$ is the Boltzmann factor for an element C with dimension r. For example, the usual two-body interaction on a bond has $r = 1$ and the lattice gauge theory has $r = 2$. $\xi_{\boldsymbol{x}}$ is a modulo-2 spin variable which takes 0 or 1, and ξ_C is defined by $\xi_C = \sum_{\boldsymbol{x}\in\partial C} \xi_{\boldsymbol{x}}$ modulo 2.

Let us define the dual Boltzmann factor for the dual element C^* by the binary Fourier transformation,

$$U^*(\lambda_{C^*}) = 2^{-\frac{1}{2}} \sum_{\xi_C=0,1} \exp(2\pi i \xi_C \lambda_{C^*}) U(\xi_C), \tag{106}$$

or conversely,

$$\begin{aligned} U(\xi_C) &= 2^{-\frac{1}{2}} \sum_{\lambda_{C^*}=0,1} \exp(2\pi i \lambda_{C^*} \xi_C) U^*(\lambda_{C^*}) \\ &= 2^{-\frac{1}{2}} \sum_{\lambda_{C^*}=0,1} \exp\left(2\pi i \lambda_{C^*} \sum_{\boldsymbol{x}\in\partial C} \xi_{\boldsymbol{x}}\right) U^*(\lambda_{C^*}) \\ &= \sum_{\lambda_{C^*}=0,1} \left(\prod_{\boldsymbol{x}\in\partial C} T(\xi_{\boldsymbol{x}}, \lambda_{C^*})\right) U^*(\lambda_{C^*}), \end{aligned} \tag{107}$$

where

$$T(\xi_{\boldsymbol{x}}, \lambda_{C^*}) \equiv 2^{-\frac{1}{2B_C}} \exp(2\pi i \lambda_{C^*} \xi_{\boldsymbol{x}}), \tag{108}$$

and B_C is the number of $(r-1)$-dimensional elements on the boundary of C.

Inserting (107) into (104) and taking the sum over $\xi_{\boldsymbol{x}}$, we obtain a modulo-2 Kronecker delta for each $\xi_{\boldsymbol{x}}$,

$$\begin{aligned}
&\sum_{\xi_{\boldsymbol{x}}=0,1} \prod_{C^*:\boldsymbol{x}\in\partial C} T(\xi_{\boldsymbol{x}}, \lambda_{C^*}) \\
&= \sum_{\xi_{\boldsymbol{x}}=0,1} \prod_{C^*:\boldsymbol{x}\in\partial C} 2^{-\frac{1}{2B_C}} \exp(2\pi i \lambda_{C^*} \xi_{\boldsymbol{x}}) \\
&= 2^{1-\sum_{C:\boldsymbol{x}\in\partial C}\frac{1}{2B_C}} \delta_{\mathrm{mod}2}\left(\sum_{C^*:\boldsymbol{x}\in\partial C} \lambda_{C^*}\right). \qquad (109)
\end{aligned}$$

It is useful to define the following symbol of a constrained sum, which stems from the Kronecker deltas in (109),

$$\sum_{\{\lambda_{C^*}=0,1\}}{}' \equiv \sum_{\{\lambda_{C^*}=0,1\}} \prod_{\boldsymbol{x}} \delta_{\mathrm{mod}2}\left(\sum_{C^*:\boldsymbol{x}\in\partial C} \lambda_{C^*}\right). \qquad (110)$$

Using this, (104) becomes

$$\mathcal{Z} = \left(\prod_{\boldsymbol{x}} 2^{1-\sum_{C:\boldsymbol{x}\in\partial C}\frac{1}{2B_C}}\right)\left(\sum_{\{\lambda_{C^*}=0,1\}}{}' \prod_{C} U^*(\lambda_{C^*})\right). \qquad (111)$$

The prefactor can be simplified as

$$\begin{aligned}
&\prod_{\boldsymbol{x}} 2^{1-\sum_{C:\boldsymbol{x}\in\partial C}\frac{1}{2B_C}} \\
&= 2^{N_{r-1}-\frac{1}{2}\sum_{\boldsymbol{x}}\sum_{C:\boldsymbol{x}\in\partial C}\frac{1}{B_C}} \\
&= 2^{N_{r-1}-\frac{N_r}{2}}, \qquad (112)
\end{aligned}$$

where the number of m-dimensional elements is denoted by N_m.

The final result is

$$\mathcal{Z} = 2^{N_{r-1}-\frac{N_r}{2}} \sum_{\{\lambda_{C^*}=0,1\}}{}' \prod_{C^*} U^*(\lambda_{C^*}). \qquad (113)$$

Here we have replaced the product over C with C^*, which are identical operations.

The partition function (104) can also be rewritten using the Boltzmann factors for the element C,

$$\begin{aligned}
\mathcal{Z} &= \sum_{\{\xi_{\boldsymbol{x}}=0,1\}} \prod_{C} U(\xi_C) \qquad (114) \\
&= 2^{\mathcal{N}_g} \sum_{\{\xi_C=0,1\}}{}' \prod_{C} U(\xi_C). \qquad (115)
\end{aligned}$$

Note that $U(\xi_C)$ in (114) is a function of $\xi_{\boldsymbol{x}}$ as in (1), but it is not in (115): It is redefined on each element C as a function of ξ_C itself. The usual sum over ξ in (114) can be written as in (115) using the definition (110) if we consider a correct mapping from the configuration space defined by $\xi_{\boldsymbol{x}}$ onto the one by ξ_C. The prefactor $2^{\mathcal{N}_g}$ in

(115) is the degree of the ground-state degeneracy. The mapping from the $\xi_{\boldsymbol{x}}$ space to the ξ_C space is not one-to-one. In (115) the ground state configuration is $\xi_C = 0$ for all C for ferromagnetic interactions and is unique, while in (114) the ground state is degenerate. Thus the mapping is $2^{\mathcal{N}_g}$ to 1. For example, $2^{\mathcal{N}_g} = 2$ for $r = 1$ because all-up and all-down states are degenerate. For $r = 2$, $2^{\mathcal{N}_g}$ is dependent on the number of sites because the model has Z_2 gauge symmetry in this case. Wegner calculated this degree of degeneracy in [5] under general conditions and the result is

$$\begin{aligned} \mathcal{N}_g &= \sum_{m=0}^{r-2} (-1)^{r-m} N_m + t_1 \qquad \text{for } r \geq 2, \\ &= 1 \qquad\qquad\qquad\qquad\quad \text{for } r = 1, \end{aligned} \tag{116}$$

where t_1 is a constant which depends on the topology of the lattice (or the boundary condition) and not on the number of elements. If we use the generalized Euler's relation for the number of lattice elements [5],

$$\sum_{m=0}^{d} (-1)^m N_m = t_2, \tag{117}$$

where t_2 is also a constant dependent on the topology, $\mathcal{N}_g$ becomes

$$\mathcal{N}_g = \sum_{m=r-1}^{d} (-1)^{m-(r-1)} N_m + t, \tag{118}$$

with t being another constant determined by t_1, t_2 and r.

We can derive the factor (118) intuitively from the difference of the numbers of spin configurations between the expressions (114) and (115). In (114) the number of independent $\xi_{\boldsymbol{x}}$'s is N_{r-1}. In (115) the number of independent ξ_C's may appear to be N_r, but we must take the number of constraints in (115) into account, which is N_{r+1} and should be subtracted from N_r. However these constraints are redundant or not independent of each other because $r+1$ dimensional element is always on the boundary of $r+2$ dimensional element. Therefore we must subtract N_{r+2} from the number of constraints N_{r+1}. However the number N_{r+2} is also redundant again and N_{r+3} must be subtracted from it, which is similar for higher dimensional elements. Then $\mathcal{N}_g$ is calculated as

$$\begin{aligned} \mathcal{N}_g &= N_{r-1} - \{N_r - (N_{r+1} - (N_{r+2} - (N_{r+3} \ldots\} \\ &= \sum_{m=r-1}^{d} (-1)^{m-(r-1)} N_m + t, \end{aligned} \tag{119}$$

which is equivalent to (118). The constant t will be the same as in (118).

Using (113), (115) and (118), we obtain the duality relation between the original and the dual partition functions,

$$\begin{aligned} & 2^{\mathcal{N}_g} \sum_{\{\xi_C = 0,1\}}{}' \prod_C U(\xi_C) \qquad (= \mathcal{Z}_{\text{orig}}\{u_1(K), u_{-1}(K)\}) \\ &= 2^{\mathcal{N}_g^* + a} \sum_{\{\lambda_{C^*} = 0,1\}}{}' \prod_{C^*} U^*(\lambda_{C^*}) \qquad \left(= 2^a \mathcal{Z}_{\text{dual}}\{u_1^*(K), u_{-1}^*(K)\}\right), \end{aligned} \tag{120}$$

where

$$\mathcal{N}_g^* = \sum_{m=0}^{r+1} (-1)^{m-(r+1)} N_m + t^*, \tag{121}$$

and t^* is also a constant which depends on the topology. a is defined by

$$a = -\mathcal{N}_g^* + N_{r-1} - \frac{N_r}{2}. \tag{122}$$

Remember that we use the modulo-2 spin variables here, while spins take the value ± 1 in the main text.

Now let us consider the self-dual case. In this case d must be even and $r = d/2$. Furthermore the number of m dimensional elements satisfies

$$N_m = N_{d-m}. \tag{123}$$

Inserting this into the Euler's relation (117), we obtain

$$\sum_{m=0}^{d/2-1} (-1)^m N_m + (-1)^{d/2} \frac{N_{d/2}}{2} = \sum_{m=d/2+1}^{d} (-1)^m N_m + (-1)^{d/2} \frac{N_{d/2}}{2} = \frac{t_2}{2}. \tag{124}$$

In addition, $\mathcal{N}_g$ and $\mathcal{N}_g^*$ become

$$\begin{aligned}
\mathcal{N}_g &= \sum_{m=d/2-1}^{d} (-1)^{m-(d/2-1)} N_m + t = N_{d/2-1} - \frac{N_{d/2}}{2} + (-1)^{d/2-1} \frac{t_2}{2} + t, \\
\mathcal{N}_g^* &= \sum_{m=0}^{d/2+1} (-1)^{m-(d/2+1)} N_m + t^* = N_{d/2+1} - \frac{N_{d/2}}{2} + (-1)^{d/2+1} \frac{t_2}{2} + t^* \\
&= N_{d/2-1} - \frac{N_{d/2}}{2} + (-1)^{d/2-1} \frac{t_2}{2} + t^* = \mathcal{N}_g + t^* - t.
\end{aligned} \tag{125}$$

Hence,

$$a = (-1)^{d/2} \frac{t_2}{2} - t^*. \tag{126}$$

Therefore a is just a trivial constant in the self-dual case and negligible in the thermodynamic limit. The factors $2^{\mathcal{N}_g}$ and $2^{\mathcal{N}_g^*}$ in (120), which differ only by a trivial constant, do not concern when we derive the transition point from the relation $u_{\pm 1}(K_c) = u_{\pm 1}^*(K_c)$.

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